SINGLE COMPTON SCATTERING

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1994, *Astrophys. Space Phys.*, 9, 1-83

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Abstract

A review of works on single Compton scattering is given. No limitations on the energies of photons and electrons are assumed. The relativistic kinetic equation, describing the scattering of the polarized radiation by a non-degenerate electron gas with known distribution function, is formulated in linear approximation. The formulae for the mean values of the Klein-Nishina cross-section and of the first and second degree of frequency of the scattered photon are given. Three types of averaging are considered: over the scattering angle, over electron directions and over electron energy distribution, which is assumed to be the relativistic Maxwellian. Using these formulae, the balance of the energy of the photon gas, the dispersion of the scattered photon frequency and the radiation pressure are investigated.

The scattering matrix describing the interaction of photons with the isotropic electron gas is obtained. It is shown that it consists of five redistribution functions (RF) on frequency, depending on the scattering angle. The RF of intensity is studied in more detail. This function is averaged over directions with the weight factors 1 and the cosine of the scattering angle. Its connection with the mean values is ascertained, and the moment equations of the zeroth and first orders are deduced.

The formulae for all quantities considered are reduced to the forms free of cancellation. In addition to the exact formulae, the asymptotic expansions of the mean values and RF are obtained in three cases: non-relativistic and ultrarelativistic electrons and the photons of small energies. In particular, the limiting expressions for the RF of intensity are found. Using these expressions, the well-known Fokker-Planck differential (on frequency) kinetic equations can be easily deduced.

1 Introduction

Compton scattering, i.e. the scattering of photons by electrons, plays a major role in astrophysical objects. This role is very important in media of low density, because the number of scattering is proportional to the first degree of the electron density $n_{\rm e}$, but the number of the other processes, which determine the formation of the continuous spectrum, for example, bremsstrahlung, is proportional to $n_{\rm e}^2$.

If the electron gas is non-relativistic, then photons of small energies (much smaller than the electron rest energy $mc^2 = 511$ keV) change energy by small increments in each scattering, and one can neglect this change. In this case the scattering is called Thomson scattering and in a sense it does not differ from the Rayleigh scattering on molecules. On the other hand, the scattering of the photon by the relativistic electron or the scattering of the hard photon by the electron even of moderate energy leads to substantial change in photon energy. In hot rarefied media the Compton scattering is the main mechanism of energy exchange between radiation and matter.

Numerous papers have been devoted to the problem of Compton scattering in astrophysical objects. If the electron gas is weakly relativistic and photons are not hard, then the changes in the photon energy are small in each scattering, but they can lead to large changes after many scatterings. In this case one can consider this process as diffusion in frequency. The investigation of the Compton scattering using a frequency diffusion equation of the Fokker-Planck type, which was deduced by Kompaneets (1956) (see also the review by Zel'dovich, 1975), gave important results on the evolution of the spectra of the microwave background radiation, X-ray sources and other relativistic objects (see Levich & Sunyaev, 1971; Zel'dovich et al., 1972; Zel'dovich & Sunyaev, 1982). Different methods of calculation of the Compton scattering in this approximation were obtained in the works by Langer et al. (1978), Illarionov et al. (1979), Lightman & Rybicki (1980), Williams (1984), Colpi (1988). The multiple Compton scattering was computed by Monte-Carlo method (Pozdnyakov et al., 1983) and in the frame of the double diffusion approximation (in the optical depth and frequency) by Sunyaev & Titarchuck (1980).

Another limiting case is the scattering of the optical, infrared or radio radiation by ultrarelativistic electrons. Such situation takes place, for example, in the jets from the active galactic nuclei, where, as assumed, the low-frequency synchrotron radiation of ultrarelativistic electrons in the weak magnetic field is scattered by the same electrons and the photon energy essentially increases. The calculation of the synchro-Compton radiation is usually based on the work by Blumenthal & Gould (1970). Among the latest papers in that direction we can mention the works by Marscher (1987) and Hughes & Miller (1991).

However, it is not always possible to reduce the problem to one of these limiting cases. In some objects electrons have moderate relativistic energies and the photon frequency changes due to the scattering are of the same order as their initial frequency. Then we must consider the general case of Compton scattering without limitations on the electron energy and photon frequency.

Another problem is Compton scattering of polarized light. The same methods were used here as those mentioned above. The interpretations of the polarimetric X-ray observation either were made on the assumption of the conservation of the photon frequency (see Loskutov & Sobolev, 1985), or the problems of studing the spectral and angular-polarization structures of the radiation field were separated. So, in the limits of the small frequencies and low temperatures of the electron gas the problem of the frequency dependence of the polarization was reduced by Sunyaev & Titarchuk (1985) to the calculation of the Rayleigh scattering of the different orders. It is possible because in the process of the multiple scattering, the final photon frequency corresponds to the definite mean number of scatterings and the scattering matrix is quite close to the Rayleigh matrix.

The scattering matrix for the case of the ultrarelativistic isotropic electron gas with a power energy spectrum was obtained by Bonometto et al. (1970).

In the present paper we review the papers devoted to the problem of the description of single Compton scattering, taking polarization into account and ignoring it. We do not limit the photon and electron energies. The main attention is paid to the formulation of the relativistic kinetic equations for the photon gas — in particular to obtaining the expressions for the absorption coefficients and scattering matrix. It is assumed that the electron gas is non-degenerate, its distribution function in momenta in all space is known and macroscopic motions of the gas are absent. In the major part of the paper we assume that the distribution function of electrons in momenta is isotropic in some frame of reference. In particular we consider the case of the relativistic Maxwellian distribution. This review is based principally on the works by Nagirner et al. (1991), Nagirner & Poutanen (1991,1993a,1993b), Poutanen (1994).

The plan of the review is as follows. Initially the definitions and notations for the values describing the photon and the electron gases are given. The kinetic equations for the Compton scattering are formulated. Then we consider different terms of the collision integral. In addition to the mean extinction coefficient of the radiation due to the scattering, we investigate the mean values of the scattering act, i.e. the mean frequency of the scattered photon, the dispersion of the frequency, the pressure of the radiation on the electron gas. Then we obtain the scattering matrix and the redistribution functions of the radiation on frequency and direction.

In addition to exact results we give the asymptotic and approximate formulae for the quantities considered and the approximate forms of the kinetic equations.

2 Relativistic kinetic equation (RKE) for photons

2.1 Description of electrons

We assume that energies of the photons and electrons may be arbitrary. Therefore, the kinetic equations which describe their interaction have to be relativistically covariant and it is necessary to introduce relativistic notations.

Let the electron momentum in the arbitrary frame of reference be $\mathbf{p}$ and the four-dimensional electron momentum $\underline{p} = \{p_0, \mathbf{p}\}$. Other four-dimensional vectors we represent by their zeroth and space (contravariant) coordinates: $\underline{a} = \{a_0, \mathbf{a}\}$. If the components of the vector $\underline{a}$ are given in some frame of reference and another frame has the same directions of the coordinate axies and moves with the dimensionless (in the unit of the speed of light c) velocity $\boldsymbol{\beta}$ relative to the first one, then vector's components in the second frame can be obtained according to the Lorentz transformation:

$$a_0' = \frac{a_0 - \boldsymbol{\beta}\cdot\mathbf{a}}{\sqrt{1-\beta^2}},\ \ \mathbf{a}' = \mathbf{a} - \frac{a_0\,\boldsymbol{\beta}}{\sqrt{1-\beta^2}} + \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)\frac{\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\mathbf{a})}{\beta^2}. \tag{2.1.1}$$

The scalar product of two vectors $\underline{a} = \{a_0, \mathbf{a}\}$ and $\underline{b} = \{b_0, \mathbf{b}\}$ we write in the form $\underline{a}\cdot\underline{b} = a_0\,b_0 - \mathbf{a}\cdot\mathbf{b}$, $\underline{a}\cdot\underline{a} = \underline{a}^2$, except that we define the "vector" product of three four-dimensional vectors by equation:

$$\underline{a}\times\underline{b}\times\underline{c} = \{(\mathbf{a}\,\mathbf{b}\,\mathbf{c}), a_0\ \mathbf{b}\times\mathbf{c} + b_0\ \mathbf{c}\times\mathbf{a} + c_0\ \mathbf{a}\times\mathbf{b}\}. \tag{2.1.2}$$

So far as we are interested only in the Compton scattering here, we assume that in region where it occurs there are only the electron and photon gases. We connect with the electron gas the fixed frame of reference E (co-moving). In this frame the mean electron momentum is equal to zero.

As was mentioned above, the kinetic equation we consider must be relativistically covariant. It means that it has the same form in all frames of reference. But it is difficult to calculate the mean values of the scattering in the arbitrary frame. Therefore, all concrete calculation is better done in the frame E.

For such calculations we use the dimensionless quantities and express the length in the Compton wavelength $\hbar/mc$, energy in mc^2, frequency in $mc^2/\hbar$. This is equivalent to choosing the system of units where the main units are the electron mass, the speed of light and the Planck constant $m = c = \hbar = 1$. In such a system the electron charge is $e = \sqrt{e^2/\hbar c} = 1/\sqrt{137.036}$, the classical electron radius $r_0 = e^2/mc^2 = e^2/\hbar c = e^2 = 1/137.036$ is equal to the fine structure constant. So, in the frame E we assume

$$\underline{p} = m\,c\{\gamma, \mathbf{z}\},\ \ \mathbf{z} = z\,\boldsymbol{\Omega},\ \ \gamma = \sqrt{1+z^2}. \tag{2.1.3}$$

Let us now turn to the description of the system of electrons, i.e. the electron gas. We denote the electron momentum distribution function $f_{\rm e}(\mathbf{p})$. Following Belyaev & Budker (1956), we assume it to be the relativistic invariant with a normalization

$$\int \underline{p}\,f_{\rm e}(\mathbf{p})\,{\rm d}^3p/p_0 = n_{\rm e}\,\underline{\beta},\ \ \underline{\beta} = \{1, \boldsymbol{\beta}\}/\sqrt{1-\beta^2}, \tag{2.1.4}$$

where $\boldsymbol{\beta}$ is the velocity of the arbitrary frame relative to E and $n_{\rm e}$ is the invariant electron density in E. In particular, the relativistic Maxwellian distribution in the arbitrary frame is defined by the formula

$$f_{\rm M}(\mathbf{p}) = \frac{n_{\rm e}}{4\,\pi\,m^2\,c\,k_{\rm B}\,T_{\rm e}\,K_2(m\,c^2/k_{\rm B}\,T_{\rm e})}\ \exp\left(-c\frac{\underline{p}\,\underline{\beta}}{k_{\rm B}\,T_{\rm e}}\right), \tag{2.1.5}$$

where K_2 is the modified Bessel function of the second kind (Macdonald function), $T_{\rm e}$ is the invariant temperature of the electron gas, $k_{\rm B}$ is Boltzmann's constant.

The isotropy of the electron distribution in the frame E means that the distribution function depends only on the energy, i.e. in that frame

$$f_{\rm e}(\mathbf{p}) = n_{\rm e}\, f_{\rm e}(\gamma)/m^3\, c^3. \tag{2.1.6}$$

The function $f_{\rm e}(\gamma)$ is normalized as follows:

$$4\,\pi \int_0^\infty z^2\, f_{\rm e}(\gamma)\, {\rm d}z = 1, \tag{2.1.7}$$

and for the Maxwellian electrons

$$f_{\rm M}(\gamma) = \frac{y}{4\,\pi\, K_2(y)}\, \exp\left(-y\,\gamma\right), \;\; y = \frac{m\,c^2}{k_{\rm B}\, T_{\rm e}}. \tag{2.1.8}$$

2.2 Description of photons

In the same manner as an electron, the photon in the arbitrary frame of reference is described by the three-dimensional momentum $\mathbf{k}$ and the four-dimensional vector of momentum $\underline{k} = \{k, \mathbf{k}\}, k = |\mathbf{k}|$. In the quasiclassical approximation the photon has the definite values of coordinates and momentum.

In order to describe the polarization state of radiation in the relativistic form, it is necessary to introduce two space-like four-dimensional vectors $\underline{e}_{(j)}(\mathbf{k})$, which satisfy the condition of orthogonality and transversality, i.e.

$$\underline{e}_{(j)}(\mathbf{k}) \cdot \underline{e}_{(j_1)}(\mathbf{k}) = -\delta_{jj_1}, \quad \underline{e}_{(j)}(\mathbf{k}) \cdot \underline{k} = 0. \tag{2.2.1}$$

Such a pair of vectors forms the relativistic polarization basis.

The polarization state of radiation is defined relative to the some polarization basis $\underline{e}_{(j)}(\mathbf{k})$ either by a polarization density matrix $\check{n}$, or by the Stokes parameters connected with each other by the equation

$$\check{n} = \left(\begin{array}{cc} n_{(11)} & n_{(12)} \\ n_{(21)} & n_{(22)} \end{array} \right) = \frac{1}{2} \left(\begin{array}{cc} n_{\rm I} + n_{\rm Q} & n_{\rm U} - i n_{\rm V} \\ n_{\rm U} + i n_{\rm V} & n_{\rm I} - n_{\rm Q} \end{array} \right). \tag{2.2.2}$$

Here $n_{\rm I}, n_{\rm Q}, n_{\rm U}, n_{\rm V}$ are dimensionless and relativistically invariant parameters relating to the usual Stokes parameters — spectral intensities — by multiplying them by $c\, k^3/4\,\pi^2\,\hbar^2$. We represent them by the vector-column $\tilde{n}$, such that $\tilde{n}^{\rm T} = (n_{\rm I}, n_{\rm Q}, n_{\rm U}, n_{\rm V})$, and name the Stokes parameters. As a rule we shall use these parameters.

If we transfer to another polarization basis $\underline{e}'_{(j)}(\mathbf{k})$, the elements of the polarization matrix and the Stokes parameters will transform. The basis vectors are connected by the angle χ, which is defined by the equations

$$\begin{array}{ll} \cos\chi = & -\underline{e}_{(1)}(\mathbf{k}) \cdot \underline{e}'_{(1)}(\mathbf{k}) = -\underline{e}_{(2)}(\mathbf{k}) \cdot \underline{e}'_{(2)}(\mathbf{k}), \\ \sin\chi = & \underline{e}_{(1)}(\mathbf{k}) \cdot \underline{e}'_{(2)}(\mathbf{k}) = -\underline{e}_{(2)}(\mathbf{k}) \cdot \underline{e}'_{(1)}(\mathbf{k}). \end{array} \tag{2.2.3}$$

Then the elements of the polarization matrix in the new basis are expressed through the elements in the old basis

$$\check{n}' = \check{f}^{-1}(\chi)\,\check{n}\,\check{f}(\chi) \tag{2.2.4}$$

by the matrix

$$\check{f}(\chi) = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix}, \quad \check{f}^{-1}(\chi) = \check{f}(-\chi) = \check{f}^{\,\mathrm{T}}(\chi). \tag{2.2.5}$$

The correspondent Stokes parameters, which are represented as a vector column, are transformed by the matrix 4×4:

$$\tilde{n}' = \hat{L}\,\tilde{n}, \tag{2.2.6}$$

where (Chandrasekhar, 1960)

$$\hat{L}(\chi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\chi & \sin 2\chi & 0 \\ 0 & -\sin 2\chi & \cos 2\chi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.2.7}$$

In the frame E we put $\mathbf{k} = m\,c\,x\,\boldsymbol{\omega}$, i.e. in this frame the photon is described by the frequency (or the momentum, that is the same in the chosen units of measurement) x and the unit vector of the photon propagation $\boldsymbol{\omega}$.

Let there exists in the considered region the marked direction, which is defined by the unit vector $\mathbf{l}$. We introduce the four-dimensional vector with components in the frame E $\underline{l} = \{0, \mathbf{l}\}$, it means that in this frame it is a pure space vector. In other systems of references its coordinates can be found according to Eq.(2.1.1). Let us connect with the vector $\mathbf{l}$ the polarization basis $\underline{e}^{\mathrm{ex}}_{(1)}(\mathbf{k})$ and $\underline{e}^{\mathrm{ex}}_{(2)}(\mathbf{k})$, which we shall call as *external*. These vectors are defined by the formulae:

$$\begin{aligned} \underline{e}^{\mathrm{ex}}_{(1)}(\mathbf{k}) &= \frac{(\underline{k}\cdot\underline{\beta})\underline{l} + (\underline{l}\cdot\underline{k})\left[\underline{k}/(\underline{k}\cdot\underline{\beta}) - \underline{\beta}\right]}{\sqrt{(\underline{k}\cdot\underline{\beta})^2 - (\underline{l}\cdot\underline{k})^2}}, \\ \underline{e}^{\mathrm{ex}}_{(2)}(\mathbf{k}) &= \frac{\underline{l}\times\underline{\beta}\times\underline{k}}{\sqrt{(\underline{k}\cdot\underline{\beta})^2 - (\underline{l}\cdot\underline{k})^2}}. \end{aligned} \tag{2.2.8}$$

These vectors form the polarization basis for the photons of momentum $\underline{k}$. It is easy to verify that they satisfy relations (2.2.1). In the frame E they are reduced to

$$\underline{e}^{\mathrm{ex}}_{(j)}(\mathbf{k}) = \{0, \mathbf{e}_j(\boldsymbol{\omega}, \mathbf{l})\}, \tag{2.2.9}$$

where ($\eta = \mathbf{l}\cdot\boldsymbol{\omega}$)

$$\mathbf{e}_1(\boldsymbol{\omega}, \mathbf{l}) = (\mathbf{l} - \eta\,\boldsymbol{\omega})/\sqrt{1-\eta^2}, \quad \mathbf{e}_2(\boldsymbol{\omega}, \mathbf{l}) = \boldsymbol{\omega}\times\mathbf{l}/\sqrt{1-\eta^2} \tag{2.2.10}$$

have the sense of the usual three-dimensional unit polarization vectors.

Introduced by equations (2.2.8) *external* basis we shall use to describe the polarization of radiation in the frame of references connected with the electron gas.

Note that the polarization matrix and vector of the Stokes parameters as well as the electron distribution function can depend on space coordinates and time. However, we shall not indicate them as arguments of these function, because the values of these variables in both sides of the kinetic equation are equal.

2.3 Description of scattering act

Let the photons of momenta $\underline{k}$ be scattered by the electrons of momenta $\underline{p}$. We denote their momenta after the scattering as $\underline{k}_1$ and $\underline{p}_1$. Then the four conservation laws have the form

$$\underline{p} + \underline{k} = \underline{p}_1 + \underline{k}_1. \tag{2.3.1}$$

Raising (2.3.1) to the second power and multiplying it by $\underline{k}$, we obtain the following consequences:

$$\underline{k} \cdot \underline{p} = \underline{k}_1 \cdot \underline{p}_1, \quad \underline{k}_1 \cdot \underline{p} = \underline{k} \cdot \underline{p}_1, \quad \underline{k}_1 \cdot (\underline{p} + \underline{k}) = \underline{k} \cdot \underline{p}. \tag{2.3.2}$$

The photon frequency after the scattering k_1 can be easily obtained from the last equality in (2.3.2):

$$k_1 = k\,\frac{p_0 - \mathbf{p}\cdot\mathbf{k}/k}{p_0 + k - (\mathbf{p}+\mathbf{k})\cdot\mathbf{k}_1/k_1}. \tag{2.3.3}$$

Let us introduce the designations for the scalar products of the electron momentum $\underline{p}$ and the photon momenta $\underline{k}$ and $\underline{k}_1$:

$$\xi = \underline{p}\cdot\underline{k}/m^2\,c^2, \quad \xi_1 = \underline{p}\cdot\underline{k}_1/m^2\,c^2, \quad q = \underline{k}\cdot\underline{k}_1/m^2\,c^2 = \xi - \xi_1. \tag{2.3.4}$$

In the last relation we take into account Eqs.(2.3.2).

In the frame E the photon of momentum $\mathbf{k}_1$ is characterized by the same quantities as $\mathbf{k}$, but with a subscript 1: $\boldsymbol{\omega}_1, \eta_1$ etc. The cosines of angles between the direction of the initial electron $\boldsymbol{\Omega}$ and the photon momenta before and after the scattering $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_1$ we denote $\zeta = \boldsymbol{\Omega}\cdot\boldsymbol{\omega}$ and $\zeta_1 = \boldsymbol{\Omega}\cdot\boldsymbol{\omega}_1$, respectively; $\mu = \boldsymbol{\omega}\cdot\boldsymbol{\omega}_1$ is the cosine of the scattering angle.

The quantities (2.3.4) are given in E by:

$$\xi = x(\gamma - z\zeta), \quad \xi_1 = x_1(\gamma - z\zeta_1), \quad q = xx_1 w = xx_1(1-\mu). \tag{2.3.5}$$

The Compton formula (2.3.3) for the frequency of the scattered photon takes the form

$$x_1 = x\frac{\gamma - z\zeta}{\gamma - z\zeta_1 + xw}. \tag{2.3.6}$$

We shall often use the designation $w = 1 - \mu$ for brevity.

As a result of the scattering, the photon energy and direction are changed. The polarization state is also changed. The elements of the density matrix before and after the scattering are connected by the two-dimesional tensor of rank 4:

$$n'_{(a_1 a'_1)} = T_{a_1 a a'_1 a'}\, n_{(aa')}. \tag{2.3.7}$$

Briefly we can write this relation by means of the tensor designation

$$\check{n}' = \overset{\diamond}{T} \; \check{n}. \tag{2.3.8}$$

The Stokes parameters are transformed in accordance with the formulae

$$\tilde{n}' = \hat{F}\, \tilde{n}. \tag{2.3.9}$$

The expressions for the elements of the matrix $\hat{F}$ through the elements of the tensor $\overset{\diamond}{T}$ are given by Nagirner & Poutanen (1993b). They are close to the expressions presented in other papers for the case when this tensor can be represented as a product of two matrices (cf. Hovenier et al., 1986; Dolginov et al., 1979).

For the scattering $\underline{k}, \underline{p} \to \underline{k}_1, \underline{p}_1$ we introduce the *internal* basis

$$\underline{e}^{\rm in}_{(1)}(\mathbf{k}) = \frac{\xi_1 \underline{k} + \xi \underline{k}_1 - q \underline{p}}{m\, c\, q\, \Delta}, \quad \underline{e}^{\rm in}_{(2)}(\mathbf{k}) = \frac{\underline{p} \times \underline{k} \times \underline{k}_1}{m^3\, c^3\, q\, \Delta}, \tag{2.3.10}$$

where

$$\Delta^2 = 2\xi\, \xi_1/q - 1 = (1+\mu_0)/(1-\mu_0), \quad \mu_0 = 1 + 1/\xi - 1/\xi_1. \tag{2.3.11}$$

In this basis the tensor of rank 4 transforming the elements of the matrix $\check{n}$ (in accordance with (2.3.7)) due to the scattering of photons on the unpolarized electrons has the simplest form (Berestetskii et al., 1971). Its non-zero elements are

$$\begin{gathered} T_{1111} = [B + 2 - 4\,(1-\mu_0^2)]/2, \quad T_{1212} = T_{2121} = (B-2)/2, \\ T_{2222} = (B+2)/2, \quad T_{1122} = T_{2211} = (B+2)\,\mu_0/2, \\ T_{1221} = T_{2112} = -(B-2)\,\mu_0/2, \end{gathered} \tag{2.3.12}$$

where

$$B = \xi/\xi_1 + \xi_1/\xi. \tag{2.3.13}$$

The elements of the matrix $\hat{F}$, which describes the transformation of the Stokes parameteres due to the scattering, in the same *internal* basis are the following:

$$\hat{F} = \begin{pmatrix} F & \mu_0^2 - 1 & 0 & 0 \\ \mu_0^2 - 1 & \mu_0^2 + 1 & 0 & 0 \\ 0 & 0 & 2\,\mu_0 & 0 \\ 0 & 0 & 0 & B\,\mu_0 \end{pmatrix}, \tag{2.3.14}$$

where

$$F = \mu_0^2 - 1 + B = \left(\frac{1}{\xi} - \frac{1}{\xi_1}\right)^2 + 2\left(\frac{1}{\xi} - \frac{1}{\xi_1}\right) + \frac{\xi}{\xi_1} + \frac{\xi_1}{\xi}. \tag{2.3.15}$$

When we consider the particle interaction, one more selected frame of references is appeared — namely, the frame where one of the interacting particles is at rest before the collision. In the case of Compton scattering such a particle can be only the electron. The frame where the initial electron is at rest, we shall name laboratory (following the terminology of the text-books by Landau & Lifshitz, 1951, and Shiff, 1956, on the quantum mechanics, but not of the modern astronomical literature).

In the laboratory frame the expressions for many quantities simplify because the electron momentum there $\underline{p} = m\,c\,\{1, 0\}$. In particular, the quantities ξ and ξ_1 are photon frequencies before and after scattering in this frame, and the quantity q can be presented in the form $q = \xi\,\xi_1\,(1-\mu_0)$, where μ_0 is the scattering angle in the same frame. Taking into account the expressions (2.3.4) written for the laboratory frame, we shall demonstrate that μ_0 from (2.3.11) is the same quantity that expressions (2.3.12) and (2.3.14) for the tensor $\overset{\diamond}{T}$ and the matrix $\hat{F}$ contain.

We note that the matrix $\hat{F}$ in the laboratory frame is the sum of the pure Rayleigh matrix and the matrix proportional to $B - 2 \geq 0$. The latter has elements only in the corners of the principal diagonal. It is obvious that the upper element produces the depolarization and the lower one leads to the additional circular polarization.

The vectors of the *internal* basis in the frame E, if the scattering is undergone by the electron at rest, are reduced to the three-dimensional:

$$\underline{e}^{\rm in}_{(1)}(\mathbf{k}) - \sqrt{r}\,\underline{k}/k \;\to\; \{0, \mathbf{e}_1(\boldsymbol{\omega}, \boldsymbol{\omega}_1)\}, \quad \underline{e}^{\rm in}_{(2)}(\mathbf{k}) \;\to\; \{0, \mathbf{e}_2(\boldsymbol{\omega}, \boldsymbol{\omega}_1)\}, \tag{2.3.16}$$

where

$$r = \frac{1+\mu}{1-\mu} = \frac{1+\mu}{w}. \tag{2.3.17}$$

The addition of a vector proportional to $\underline{k}$ to the first vector has no influence on the description of the polarization state. The three-dimensional *internal* vectors are expressed by similar formulae to those descibing the *external* vectors (2.2.10):

$$\mathbf{e}_1(\boldsymbol{\omega}, \boldsymbol{\omega}_1) = (\boldsymbol{\omega}_1 - \mu\,\boldsymbol{\omega})/\sqrt{1-\mu^2}, \quad \mathbf{e}_2(\boldsymbol{\omega}, \boldsymbol{\omega}_1) = \boldsymbol{\omega} \times \boldsymbol{\omega}_1/\sqrt{1-\mu^2}. \tag{2.3.18}$$

These two vectors together with $\boldsymbol{\omega}$ form the three-dimensional basis. In this basis the direction of the electron propagation in the frame E is defined by the cosine ζ and the azimuth Φ:

$$\mathbf{\Omega} = \sqrt{1-\zeta^2}\,[\cos\Phi\;\mathbf{e}_1(\boldsymbol{\omega}, \boldsymbol{\omega}_1) + \sin\Phi\;\mathbf{e}_2(\boldsymbol{\omega}, \boldsymbol{\omega}_1)] + \zeta\,\boldsymbol{\omega}. \tag{2.3.19}$$

Now we can formulate the kinetic equation for Compton scattering.

2.4 Kinetic equation

The relativistic kinetic equation describing the scattering of the polarized radiation by the electron gas in terms of the polarization density matrices in the linear approximation in the arbitrary frame of reference may be written as:

$$\underline{k} \cdot \underline{\nabla}\,\check{n} = -\frac{r_0^2}{2} m^2 c^2 \int \frac{{\rm d}^3 p}{p_0} \frac{{\rm d}^3 p_1}{p_{10}} \frac{{\rm d}^3 k_1}{k_1} \delta(\underline{p} + \underline{k} - \underline{p}_1 - \underline{k}_1)\,[f_{\rm e}(\mathbf{p}) F\,\check{n} - f_{\rm e}(\mathbf{p}_1)\check{n}^{\rm s}_1]\,. \tag{2.4.1}$$

Here for the matrix describing the scattered photon we introduce the designation

$$\check{n}^{\rm s}_1 = \check{f}(-\chi)\;\overset{\diamond}{T}\;(\check{f}(\chi_1)\,\check{n}_1\,\check{f}(-\chi_1))\,\check{f}(\chi), \quad \check{n}_1 = \check{n}(\mathbf{k}_1). \tag{2.4.2}$$

The Dirac δ-function in the integrand enforces the conservation laws. The matrices $\check{f}$ transform the density matrices from the *external* polarization basis to the *internal* one and back.

Equation (2.4.1) can be derived from the quantum Liouville equation by the Bogolubov method in the manner that was used by Silin (1971) for another problem. From Eq.(2.4.1) we obtain the equation for the Stokes vector:

$$\underline{k}\cdot\underline{\nabla}\,\tilde{n} = -\frac{r_0^2}{2}m^2c^2\int\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p_1}{p_{10}}\frac{\mathrm{d}^3k_1}{k_1}\delta(\underline{p}+\underline{k}-\underline{p}_1-\underline{k}_1)\left[f_{\mathrm{e}}(\mathbf{p})\,F\,\tilde{n}-f_{\mathrm{e}}(\mathbf{p}_1)\tilde{n}_1^{\mathrm{s}}\right], \quad (2.4.3)$$

where, as in Eq.(2.4.1), we denote

$$\tilde{n}=\tilde{n}(\mathbf{k}),\ \ \tilde{n}_1=\tilde{n}(\mathbf{k}_1),\ \ \tilde{n}_1^{\mathrm{s}}=\hat{L}(-\chi)\,\hat{F}\,\hat{L}(\chi_1)\tilde{n}_1. \quad (2.4.4)$$

In the last term in the square brackets of equations (2.4.1) and (2.4.3) the scalar quantity F is the Klein-Nishina cross-section, which is not a matrix. This fact is the consequence of the isotropy of the medium.

By means of the designation

$$\hat{R}(\mathbf{k}_1\to\mathbf{k}) = \frac{3}{16\pi}\frac{m^3c^3}{n_{\mathrm{e}}}\int\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p_1}{p_{10}}\delta(\underline{p}_1+\underline{k}_1-\underline{p}-\underline{k})f_{\mathrm{e}}(\mathbf{p}_1)\hat{L}(-\chi)\hat{F}\hat{L}(\chi_1), \quad (2.4.5)$$

we rewrite Eq.(2.4.3) in the following form

$$\underline{k}\cdot\underline{\nabla}\,\tilde{n} = -k\,\sigma_{\mathrm{T}}\,\overline{s}_0(k)\,n_{\mathrm{e}}\,\tilde{n}+\sigma_{\mathrm{T}}\frac{n_{\mathrm{e}}}{m\,c}\int\frac{\mathrm{d}^3k_1}{k_1}\,\hat{R}(\mathbf{k}_1\to\mathbf{k})\,\tilde{n}_1, \quad (2.4.6)$$

where $\sigma_{\mathrm{T}}=8\,\pi\,r_0^2/3$ is the Thomson cross-section and the product

$$n_{\mathrm{e}}\,\sigma_{\mathrm{T}}\,\overline{s}_0(k) = \frac{r_0^2}{2\,k}m^2c^2\int\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p_1}{p_{10}}\frac{\mathrm{d}^3k_1}{k_1}\,f_{\mathrm{e}}(\mathbf{p})\,F\,\delta(\underline{p}_1+\underline{k}_1-\underline{p}-\underline{k}) \quad (2.4.7)$$

is the extinction coefficient of radiation due to the Compton scattering. Note that $k\,\overline{s}_0(k)$ and the elements of the matrix $\hat{R}$ are the relativistic invariants. Besides, $\hat{R}$ and $\overline{s}_0(k)$ are dimensionless, which directly follows from their definitions.

Now let us consider the terms of equation (2.4.3). First, the term connected with the extinction of radiation and then the term describing the scattering.

3 Mean values

3.1 Mean extinction coefficient and mean degrees of frequency

As we can see from the definition (2.4.7), the extinction coefficient due to the Compton scattering is proportional to the mean cross-section (in units of the Thomson cross-section)

$$\overline{s}_0(k) = \frac{3}{16\,\pi}\frac{m^2\,c^2}{k\,n_{\mathrm{e}}}\int\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p_1}{p_{10}}\frac{\mathrm{d}^3k_1}{k_1}f_{\mathrm{e}}(\mathbf{p})\,F\,\delta(\underline{p}_1+\underline{k}_1-\underline{p}-\underline{k}). \quad (3.1.1)$$

Calculating the integral over $\mathbf{p}_1$ and using the identity

$$\delta(p_{10}+k_1-p_0-k)=\delta(\underline{k}_1\cdot(\underline{p}+\underline{k})-\underline{p}\cdot\underline{k})\,p_{10}, \tag{3.1.2}$$

which can be deduced directly from (2.3.2) and (2.3.3), we obtain the following expression for the dimensionless mean scattering cross-section:

$$\overline{s}_0(k)=\frac{3\,m^2\,c^2}{16\,\pi\,n_{\rm e}\,k}\int\frac{{\rm d}^3p}{p_0}\,\frac{{\rm d}^3k_1}{k_1}f_{\rm e}(\mathbf{p})\,F\,\delta(\underline{k}_1\cdot(\underline{p}+\underline{k})-\underline{p}\cdot\underline{k}). \tag{3.1.3}$$

In the frame E, where the electron distribution is isotropic, the equality (3.1.3) can be presented in the form

$$\begin{aligned}\overline{s}_0(x) &= \frac{3}{16\pi x}\int_0^\infty\frac{z^2{\rm d}z}{\gamma}f_{\rm e}(\gamma)\int{\rm d}^2\Omega\int{\rm d}^2\omega_1\int_0^\infty x_1\,{\rm d}x_1F\\ &\times\ \delta(x_1[\gamma+x-\boldsymbol{\omega}_1\cdot(z\boldsymbol{\Omega}+x\boldsymbol{\omega})]-\xi).\end{aligned} \tag{3.1.4}$$

By making the change of variables in the external integral and calculating the internal integral, one obtains the following expression

$$\begin{aligned}\overline{s}_0(x) &= \frac{3}{16\,\pi\,x}\int_1^\infty z{\rm d}\gamma\,f_{\rm e}(\gamma)\int\frac{{\rm d}^2\Omega}{\xi}\int{\rm d}^2\omega_1\,x_1^2\,F\\ &= \frac{1}{x}\int_1^\infty z{\rm d}\gamma\,f_{\rm e}(\gamma)\int{\rm d}^2\Omega\,\xi\,s_0(\xi),\end{aligned} \tag{3.1.5}$$

where now the frequency x_1 is defined by formula (2.3.6) and

$$s_0(\xi)=\frac{3}{16\,\pi\,\xi^2}\int x_1^2\,F\,{\rm d}^2\omega_1. \tag{3.1.6}$$

In addition to the mean extinction coefficient, let us define the mean frequencies of the scattered photon. We will consider three types of averaging: at first, over the directions of the outgoing photon, then over directions of the incoming electrons and, finally, over electron energy distribution.

So, at the beginning, we introduce the total cross-section of the Compton scattering (in units of the Thomson cross-section) and the degrees of frequency averaged over directions

$$<x_1^j>\ s_0(\xi)=\frac{3}{16\,\pi\,\xi^2}\int x_1^{j+2}\,F\,{\rm d}^2\omega_1. \tag{3.1.7}$$

Now we define the mean values averaged over electron direction

$$\Psi_j(x,\gamma)=\frac{1}{4\,\pi\,\gamma\,x^{j+1}}\int{\rm d}^2\Omega<x_1^j>\ \xi\,s_0(\xi) \tag{3.1.8}$$

and over electron energies

$$\overline{x_1^j}\,\overline{s}_0(x)=4\,\pi\,x^j\int_0^\infty z^2\,f_{\rm M}(\gamma)\,{\rm d}z\,\Psi_j(x,\gamma). \tag{3.1.9}$$

Let us consider these stages in turn.

3.2 Averaging over photon directions

The total scattering cross-section $\sigma_{\mathrm{T}}\, s_0(\xi)$ is a relativistic invariant because the quantities F, ξ and $x_1^2\,\mathrm{d}^2\omega_1$ are invariants, so it depends only on ξ. The mean frequencies are not invariants. They depend on the frequency of the initial photon, the electron energy and the angle between their momenta. Therefore, it is important what frame of references we choose to calculate the mean frequencies. We shall give all results in the frame E. To perform such calculation we have to introduce concrete angle variables, which characterize the vectors' directions.

We fix the electron and photon momenta before the scattering $z\,\mathbf{\Omega}$ and $x\,\boldsymbol{\omega}$ and form the three-dimensional basis by these vectors as in (2.2.10) and (2.3.18):

$$\mathbf{e}_1(\boldsymbol{\omega},\mathbf{\Omega}) = \frac{\mathbf{\Omega}-\zeta\,\boldsymbol{\omega}}{\sqrt{1-\zeta^2}}, \quad \mathbf{e}_2(\boldsymbol{\omega},\mathbf{\Omega}) = \frac{\boldsymbol{\omega}\times\mathbf{\Omega}}{\sqrt{1-\zeta^2}}, \quad \mathbf{e}_3 = \boldsymbol{\omega}, \tag{3.2.1}$$

where $\zeta = \boldsymbol{\omega}\cdot\mathbf{\Omega}$, therefore $\mathbf{\Omega} = \sqrt{1-\zeta^2}\,\mathbf{e}_1(\boldsymbol{\omega},\mathbf{\Omega}) + \zeta\,\mathbf{e}_2(\boldsymbol{\omega},\mathbf{\Omega})$. Then we decompose the unit vector of the scattered photon momentum in these unit vectors:

$$\boldsymbol{\omega}_1 = \sqrt{1-\mu^2}\,(\cos\phi\,\mathbf{e}_1(\boldsymbol{\omega},\mathbf{\Omega}) + \sin\phi\,\mathbf{e}_2(\boldsymbol{\omega},\mathbf{\Omega})) + \mu\,\boldsymbol{\omega}. \tag{3.2.2}$$

The cosine

$$\zeta_1 = \mathbf{\Omega}\cdot\boldsymbol{\omega}_1 = \mu\,\zeta + \sqrt{1-\mu^2}\,\sqrt{1-\zeta^2}\,\cos\phi. \tag{3.2.3}$$

All quantities in the integrals (3.1.7) — (3.1.9) are expressed in terms of the introduced variables: x_1, by the equality (2.3.6), ξ and ξ_1, in accordance with Eq.(2.3.5); $\mathrm{d}^2\omega_1 = \mathrm{d}\mu\,\mathrm{d}\phi$. But the calculation in these variables is very difficult and we introduce new angles in the laboratory frame.

The corresponding quantities in the laboratory frame will be marked 0. The passage to this frame from the frame E is performed by the Lorentz transformation with the velocity $\boldsymbol{\beta} = z\,\mathbf{\Omega}/\gamma$. The frequencies x and x_1 transfer to ξ and ξ_1, and the scattering angle in this frame μ_0 was already used for writing expressions (2.3.12) and (2.3.14). The inverse passage is carried out with velocity $-\boldsymbol{\beta}$; therefore, all inverse transformations can be obtained by making the change $z\to -z$. In particular, the inversion of (2.3.5) is the equalities

$$x = \xi\,(\gamma + z\,\zeta_0), \quad x_1 = \xi_1\,(\gamma + z\,\zeta_1^0). \tag{3.2.4}$$

As a result of the application of the Lorentz transformation, we obtain the vectors $\boldsymbol{\omega}^0$ and $\boldsymbol{\omega}_1^0$, but the vector $\mathbf{\Omega}$ does not change. Two vectors $\mathbf{\Omega}$ and $\boldsymbol{\omega}^0$ give rise to the basis $\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{e}_3^0 = \boldsymbol{\omega}^0$. The unit vectors are defined by the same formulae as (3.2.1), but $\boldsymbol{\omega}$ and ζ are replaced by $\boldsymbol{\omega}^0$ and $\zeta_0 = \mathbf{\Omega}\cdot\boldsymbol{\omega}^0$. The formulae for the $\mathbf{\Omega}, \boldsymbol{\omega}_1^0$ and ζ_1^0 can be written if we add index 0. For example,

$$\zeta_1^0 = \mathbf{\Omega}\cdot\boldsymbol{\omega}_1^0 = \mu_0\,\zeta_0 + \sqrt{1-\mu_0^2}\,\sqrt{1-\zeta_0^2}\,\cos\phi_0. \tag{3.2.5}$$

The invariant q was already written in both frames — namely:

$$q = x\,x_1\,(1-\mu) = \xi\,\xi_1\,(1-\mu_0). \tag{3.2.6}$$

As it is mentioned above, this equation agrees with the definition of μ_0 and with the consequence of the conservation laws $\xi = \xi_1 + q = \xi_1 \, [1 + \xi \, (1 - \mu_0)]$. The simple relation can be obtained between the projections of the photon momentum in both frames on the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_2^0$, because these vectors, as perpendicular to $\mathbf{\Omega}$, coincide:

$$x_1 \sqrt{1 - \mu^2} \, \sin \phi = \xi_1 \sqrt{1 - \mu_0^2} \, \sin \phi_0. \tag{3.2.7}$$

The quantities μ_0 and ξ_1 are connected, and therefore we can choose ξ_1 and ϕ_0 as new variables. The relations between them and μ, ϕ are defined by the formulae:

$$\xi_1 = \xi \, \frac{\gamma - z \, \zeta_1}{\gamma - z \, \zeta_1 + x \, w}, \quad \tan \phi_0 = \frac{\xi \, \sin \phi}{\xi \, \cos \phi - x \, z \sqrt{1 - \zeta^2} / r}, \tag{3.2.8}$$

where r is given by Eq. (2.3.17). The element of surface $x_1^2 \, \mathrm{d}^2 \omega_1$ in the new variables is $\mathrm{d}\xi_1 \, \mathrm{d}\phi_0$. The expression for the frequency x_1 can be obtained from (3.2.4) and (3.2.5), and takes the form

$$x_1 = \xi_1 \, [\gamma + z \, (\mu_0 \, \zeta_0 + \sqrt{1 - \mu_0^2} \sqrt{1 - \zeta_0^2} \, \cos \phi_0)]. \tag{3.2.9}$$

The quantities μ_0 and $z \, \zeta_0$ can be excluded, using Eqs.(2.3.11) and (3.2.4).

We note that the quantity F does not depend on ϕ_0 and in powers of x_1 the quantities $\sqrt{1 - \mu_0^2}$ and $\cos \phi_0$ enter only in the form of product. The integrals over ϕ of the odd powers of $\cos \phi_0$ are equal to zero; therefore, all square roots $\sqrt{1 - \mu_0^2}$ disappear. The limits of integration over ξ_1 are defined from Eq. (2.3.11). So, ξ_1 changes from $\xi/(1 + 2\,\xi)$ for $\mu_0 = -1$ to ξ for $\mu_0 = 1$.

Thus, definition (3.1.7) can be rewritten in new variables in the following form:

$$< x_1^j > \; s_0(\xi) = \frac{3}{16 \, \pi \, \xi^2} \int_{\xi/(1+2\,\xi)}^{\xi} F \, \mathrm{d}\xi_1 \int_0^{2\,\pi} x_1^j \mathrm{d}\phi_0. \tag{3.2.10}$$

We consider here the total cross-section and the first two powers of frequency. To write the results of calculation, it is convenient to introduce the functions

$$s_j(\xi) = \frac{3}{8 \, \xi^{j+2}} \int_{\xi/(1+2\,\xi)}^{\xi} \xi_1^j F \, \mathrm{d}\xi_1. \tag{3.2.11}$$

For the total cross-section one obtains the well-known formula (cf. Berestetskii et al., 1971):

$$s_0(\xi) = 3 \, [4 + (\xi - 2 - 2/\xi) \, l_\xi + 2 \, \xi^2 \, (1 + \xi) \, R_\xi^2] \, / 8 \, \xi^2. \tag{3.2.12}$$

Hereafter

$$l_\xi = \ln (1 + 2\,\xi), \quad R_\xi = 1/(1 + 2\,\xi). \tag{3.2.13}$$

In the same designations

$$s_1(\xi) = 3 \, [l_\xi + 4 \, \xi^2/3 - 3 \, \xi/2 - \xi \, R_\xi \, (1/2 + \xi \, R_\xi^2/3)] \, / 8 \, \xi^3, \tag{3.2.14}$$
$$s_2(\xi) = (1 + 4 \, \xi + 7 \, \xi^2 + 9 \, \xi^3/2) \, R_\xi^4. \tag{3.2.15}$$

Then for the mean frequency of the scattered photon we obtain

$$< x_1 > \; s_0(\xi) = \xi\,[\gamma\, S_1(\xi) + x\, S_2(\xi)], \tag{3.2.16}$$

and for the mean square of frequency

$$< x_1^2 > \; s_0(\xi) = \gamma^2\, \xi^2\, S_4(\xi) - \gamma\, x\, \xi\, S_5(\xi) + x^2\, S_6(\xi) - \xi^2\, S_7(\xi). \tag{3.2.17}$$

All functions $S_j(\xi)$ are elementary and are expressed by means of introduced functions (3.2.11). In addition, we define them in such a manner that they do not become zero if $\xi = 0$:

$$\begin{gathered} S_1(\xi) = [s_0(\xi) - s_1(\xi)]/\xi, \quad S_2(\xi) = [s_1(\xi) - S_1(\xi)]/\xi, \\ S_3(\xi) = [s_1(\xi) - s_2(\xi)]/\xi, \quad S_4(\xi) = [S_1(\xi) - S_3(\xi)]/\xi, \\ S_5(\xi) = 3\, S_4(\xi) - 4\, S_3(\xi), \quad S_7(\xi) = S_3(\xi) - S_4(\xi)/2, \\ S_6(\xi) = s_2(\xi) - 3\, S_7(\xi). \end{gathered} \tag{3.2.18}$$

The expressions for $< x_1 >$ were obtained in the papers by Guilbert (1986); Haruthyunian et al. (1988); Shestakov et al. (1988); Barbosa (1982). The special cases of Eq.(3.2.17) were considered by Barbosa (1982).

Now we obtain the values averaged over the electron directions.

3.3 Averaging over electron directions

Since the medium in which the scattering takes place, is isotropic, the functions Ψ_j defined by Eqs. (3.1.8) and (3.1.9) and the mean values of frequency x_1 depend only on the photon frequency x (and on the electron characteristics), but do not depend on the direction of its momentum $\boldsymbol{\omega}$.

As previously, we limit ourselves by $j = 0, 1, 2$. As the variables of integration in (3.1.8), we choose the cosine ζ of the angle between $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ and the corresponding azimuth. In accordance with Eq. (2.3.5) $\xi = x\,(\gamma - z\,\zeta)$ and $< x_1^j >$ depend only on the fixed values of x, γ and ξ. The integral over azimuth can be replaced by $2\,\pi$ and we can choose ξ as the variable of integration instead of ζ. Then

$$\Psi_j(x,\gamma) = \frac{1}{2\,\gamma\, z\, x^{j+2}} \int_{x\,(\gamma - z)}^{x\,(\gamma+z)} < x_1^j > \; \xi\, s_0(\xi)\, \mathrm{d}\xi. \tag{3.3.1}$$

Introduced functions may be expressed in terms of the functions of one variable, namely

$$\begin{aligned} \psi_{ij}(\xi) &= \frac{i+1}{\xi^{i+1}} \int_0^{\xi} \xi'^i\, s_j(\xi')\, \mathrm{d}\xi', \quad i > -1, \\ \psi_{-1j}(\xi) &= \frac{1}{\xi} \int_0^{\xi} [1 - s_j(\xi')] \frac{\mathrm{d}\xi'}{\xi'}. \end{aligned} \tag{3.3.2}$$

For $j = 0$ it is obvious; for $j = 1, 2$ such a possibility can be found out after the substitution of (3.2.16), (3.2.17) and expressions (3.2.18) for S_j to (3.3.1). Let us write the expressions obtained as result of such substitution in the following form:

$$
\begin{aligned}
\Psi_0(x,\gamma) &= \frac{1}{2\,\gamma\,z}\frac{u^2}{2}\,\psi_{10}(x\,u)\Big|_{\gamma-z}^{\gamma+z}, \\
\Psi_1(x,\gamma) &= \frac{1}{2\,\gamma\,z}\frac{u^3}{3}\,\left[\gamma\Psi_{21}(x\,u) + x\,\Psi_{22}(x\,u)\right]\Big|_{\gamma-z}^{\gamma+z}, \\
\Psi_2(x,\gamma) &= \frac{1}{2\,\gamma\,z}\left[\gamma^2\frac{u^4}{4}\Psi_{34}(x\,u) - \gamma\,\frac{u^3}{3}\Psi_{25}(x\,u) + \frac{u^2}{2}\Psi_{16}(x\,u) - \frac{u^4}{4}\Psi_{37}(x\,u)\right]\Big|_{\gamma-z}^{\gamma+z},
\end{aligned}
\tag{3.3.3}
$$

where a new set of functions with two subscripts is introduced :

$$
\Psi_{ij}(\xi) = \frac{i+1}{\xi^{i+1}}\int_0^{\xi} \xi'^{i}\,S_j(\xi')\,\mathrm{d}\xi'. \tag{3.3.4}
$$

Since functions S_j are expressed through s_j, functions Ψ_{ij} can be represented through ψ_{ij}:

$$
\begin{gathered}
\Psi_{11} = 2\,(\psi_{00} - \psi_{01})/\xi, \;\; \Psi_{13} = 2\,(\psi_{01} - \psi_{02})/\xi, \\
\Psi_{14} = 2\,(\psi_{-11} - \psi_{-10} - \psi_{-12})/\xi, \\
\Psi_{16} = \psi_{12} - 3\,\Psi_{13} + 3\,\Psi_{14}/2, \\
\Psi_{21} = 3\,(\psi_{10} - \psi_{11})/2\xi, \;\; \Psi_{23} = 3\,(\psi_{11} - \psi_{12})/2\xi, \\
\Psi_{22} = 3\,(\psi_{11} - \Psi_{11})/2\xi, \;\; \Psi_{24} = 3\,(\Psi_{11} - \Psi_{13})/2\xi, \\
\Psi_{33} = 4\,(\psi_{21} - \psi_{22})/3\xi, \;\; \Psi_{34} = 4\,(\Psi_{21} - \Psi_{23})/3\xi, \\
\Psi_{25} = 3\,\Psi_{24} - 4\,\Psi_{23}, \;\; \Psi_{37} = \Psi_{33} - \Psi_{34}/2.
\end{gathered}
\tag{3.3.5}
$$

In these formulae we omit the argument ξ. These functions do not vanish at $\xi = 0$.

In Eqs. (3.3.3) we must take the difference of the values on $u = \gamma + z$ and $u = \gamma - z$. Thus, to calculate the functions Ψ_0, Ψ_1 and Ψ_2 one must know 11 functions ψ_{ij}.

All necessary functions are elementary or include one more function (already used in the paper by Haruthyunian et al. (1988)):

$$
g(\xi) = \int_0^{\xi} \ln{(1 + 2\,\xi')}\,\mathrm{d}\xi'/\xi'. \tag{3.3.6}
$$

It is possible to write a relation between the values of this function on $\xi < 1/2$ and $\xi > 1/2$. Let us define for that the auxiliary function for $\xi \le 1$

$$
g_*(\xi) = g(\xi/2) = \int_0^{\xi} \ln{(1 + \xi')}\mathrm{d}\xi'/\xi'. \tag{3.3.7}
$$

It can be presented by series

$$
g_*(\xi) = \begin{cases} \displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\xi^n/n^2, & 0 \le \xi \le \xi_* < 1, \\ \displaystyle\frac{\pi^2}{12} + \ln 2\,\ln\xi + \sum_{k=0}^{\infty}\frac{(1-\xi)^{k+2}}{k+2}\sum_{m=1}^{k+1}\frac{1}{2^m m}, & \xi_* \le \xi \le 1. \end{cases} \tag{3.3.8}
$$

As ξ_* we can take 0.8 — 0.9. Then

$$g(\xi) = \begin{cases} g_*(2\,\xi), & 0 \le \xi \le 1/2, \\ \pi^2/12, & \xi = 1/2, \\ \pi/6 + \ln^2(2\,\xi)/2 - g_*(1/2\,\xi), & \xi \ge 1/2. \end{cases} \tag{3.3.9}$$

Other methods of calculation of the function $g(\xi)$ are given by Luke (1975).

The expressions for the functions ψ_{ij} are the following:

$$\begin{aligned}
\psi_{-10}(\xi) &= \frac{1}{8\,\xi}\left[\frac{4}{\xi^2} + \frac{2}{\xi} + \left(8 + \frac{3}{\xi} - \frac{3}{\xi^2} - \frac{2}{\xi^3}\right) l_\xi - 3\,R_\xi - \frac{11}{3}\right], \\
\psi_{-11}(\xi) &= \frac{1}{8\,\xi}\left[\frac{7}{3} + \frac{2}{\xi} - \frac{2}{\xi^2} + \left(8 + \frac{1}{\xi^3}\right) l_\xi - 4\,R_\xi - R_\xi^2\right], \\
\psi_{-12}(\xi) &= \frac{1}{16\,\xi}\left[11 + 16\,l_\xi - 7\,R_\xi - 3\,R_\xi^2 - R_\xi^3\right], \\
\psi_{00}(\xi) &= \frac{3}{8\,\xi}\left[g(\xi) - \frac{2}{\xi} + \left(\frac{1}{2} + \frac{2}{\xi} + \frac{1}{\xi^2}\right) l_\xi - \frac{1}{2}\,R_\xi - R_\xi^2\right], \\
\psi_{01}(\xi) &= \frac{3}{8\,\xi}\left[\frac{1}{\xi} - \frac{1}{2} + \left(\frac{4}{3} - \frac{1}{2\,\xi^2}\right) l_\xi - \frac{1}{3}\,R_\xi - \frac{1}{6}\,R_\xi^2\right], \\
\psi_{02}(\xi) &= \frac{1}{32\,\xi}\left[\frac{7}{2} + 9\,l_\xi - R_\xi - \frac{3}{2}\,R_\xi^2 - R_\xi^3\right], \\
\psi_{10}(\xi) &= \frac{3}{4\,\xi^2}\left[\left(\xi + \frac{9}{2} + \frac{2}{\xi}\right) l_\xi - 4 - \xi + \xi^2\,R_\xi - 2\,g(\xi)\right], \\
\psi_{11}(\xi) &= \frac{1}{4\,\xi^2}\left[6 + 4\,\xi - 3\left(\frac{3}{2} + \frac{1}{\xi}\right) l_\xi - \xi\,(1+\xi)\,R_\xi^2\right], \\
\psi_{12}(\xi) &= \frac{1}{16\,\xi^2}\left[\frac{1}{2} + 9\,\xi - 4\,l_\xi - R_\xi + \frac{1}{2}\,R_\xi^3\right], \\
\psi_{21}(\xi) &= \frac{9}{8\,\xi^3}\left[g(\xi) + \frac{2}{3}\,\xi^2 - \frac{3}{2}\,\xi - \frac{1}{4}\,l_\xi - \frac{1}{6}\,\xi^2\,R_\xi^2\right], \\
\psi_{22}(\xi) &= \frac{3}{64\,\xi^3}\left[9\,\xi^2 - 8\,\xi + 5\,l_\xi - \xi\,(2 + 11\xi + 10\,\xi^2)\,R_\xi^3\right].
\end{aligned} \tag{3.3.10}$$

To prevent the loss of accuracy if ξ is very small, we can use the series which we shall obtain below.

3.4 Averaging over a Maxwellian distribution

Let us now average our quantities over the relativistic Maxwellian distribution of electron energies, i.e. calculate the integrals

$$\overline{x_1^j}\,\overline{s}_0(x) = 4\,\pi\,x^j \int_1^\infty z\,\gamma\,f_{\rm M}(\gamma)\,{\rm d}\gamma\,\Psi_j(x,\gamma). \tag{3.4.1}$$

The mean frequencies $\overline{x_1^j}$ depend on the frequency of the initial photon and also on the temperature of the electron gas through the paramater y (formula 2.1.8). The dependence on y we do not show explicitly.

The preliminary study of integrals like (3.4.1) shows that it is better to use γ as the variable of integration, but not z. In addition, it is convenient to integrate by parts. Then for $j = 0, 1, 2$ we obtain, respectively,

$$\begin{aligned}
\overline{s}_0(x) &= \frac{1}{2K_2(y)} \int_1^\infty \exp\left(-y\,\gamma\right) \frac{\mathrm{d}\gamma}{z}\, u^2\, s_0(x\,u)\Big|_{\gamma-z}^{\gamma+z} (+), \\
\frac{\overline{x_1}}{x}\, \overline{s}_0(x) &= \frac{1}{2K_2(y)} \int_1^\infty \exp\left(-y\,\gamma\right) \frac{\mathrm{d}\gamma}{z} \left[\left(\gamma + \frac{1}{y}\right) u^3\, S_1(x\,u)\right. \\
&+ \left. x\, u^3\, S_2(x\,u)\right]\Big|_{\gamma-z}^{\gamma+z} (+), \qquad (3.4.2)\\
\frac{\overline{x_1^2}}{x^2}\, \overline{s}_0(x) &= \frac{1}{2K_2(y)} \int_1^\infty \exp\left(-y\,\gamma\right) \frac{\mathrm{d}\gamma}{z} \left[\left(\gamma^2 + \frac{2\,\gamma}{y} + \frac{2}{y^2}\right) u^4\, S_4(xu)\right. \\
&- \left.\left(\gamma + \frac{1}{y}\right) u^3\, S_5(x\,u) + u^2\, S_6(x\,u) - u^4\, S_7(x\,u)\right]\Big|_{\gamma-z}^{\gamma+z} (+).
\end{aligned}$$

Here the sign (+) at the end of the expressions (3.4.2) means that it is necessary to take a sum of results after the substitution $u = \gamma + z$ and $u = \gamma - z$, but not the difference, as in (3.3.3).

For small and large values of x we can obtain asymptotic expansions of the mean values, which we present at the next section.

4 Asymptotic behaviour of mean values

4.1 Functions $s_j(\xi)$ and $S_j(\xi)$

All function s and S can be expanded to the series, which converge in the region $\xi < 1/2$. It is easy to show that

$$s_j(\xi) = \sum_{n=0}^{\infty} a_{jn}\, (-2\,\xi)^n, \quad S_j(\xi) = \sum_{n=0}^{\infty} A_{jn}\, (-2\,\xi)^n, \qquad (4.1.1)$$

where

$$\begin{gathered}
a_{0n} = 3\,[n + 2 + 2/(n+1) + 8/(n+2) - 16/(n+3)]/8, \qquad (4.1.2)\\
a_{1n} = [n\,(n+5) + 24/(n+3)]/8, \quad a_{2n} = (n^3 + 9\,n^2 + 22\,n + 32)/32.
\end{gathered}$$

Using Eqs.(3.2.18), one can obtain the expressions for the coefficients A through a:

$$\begin{gathered}
A_{1\,n} = 2\,(a_{1\,n+1} - a_{0\,n+1}), \quad A_{2\,n} = 2\,(A_{1\,n+1} - a_{1\,n+1}), \\
A_{3\,n} = 2\,(a_{2\,n+1} - a_{1\,n+1}), \quad A_{4\,n} = 2\,(A_{3\,n+1} - A_{1\,n+1}), \qquad (4.1.3)\\
A_{5\,n} = 3\,A_{4\,n} - 4A_{3\,n}, \quad A_{7\,n} = A_{3\,n} - A_{4\,n}/2, \quad A_{6\,n} = a_{2\,n} - 3\,A_{7\,n}.
\end{gathered}$$

There is no losses of accuracy when calculating functions s_2, S_5, S_6, S_7. We note that it is not necessary to expand the whole function in a series. The terms which lead to loss of significance, have the form $\ln(1+2\,\xi)/\xi^n$, $n = 1\div 5$. Therefore, it is sufficient to subtract from $\ln(1+2\,\xi)$ the first $n-1$ terms of the expansion and to find the remainder by the series.

It is no difficult to obtain the expansions of our functions in series for large ξ. But these expansions begin from the terms of order $1/\xi^4$, and we have to write the initial terms for every function separately. Thus,

$$s_0(\xi) = \frac{3}{8\,\xi^2}\left[\left(\xi - 2 - \frac{2}{\xi}\right)\ln 2\,\xi + \frac{\xi}{2} + \frac{9}{2} - \frac{5}{4\,\xi} + \sum_{n=0}^{\infty}\left(-\frac{1}{2\,\xi}\right)^{n+2} b_{0n}\right], \quad (4.1.4)$$

$$s_1(\xi) = \frac{3}{8\,\xi^2}\left[\frac{4}{3}\,\xi - \frac{3}{2} + \frac{1}{\xi}\,\ln 2\,\xi - \frac{1}{4\,\xi} - \sum_{n=0}^{\infty}\left(-\frac{1}{2\,\xi}\right)^{n+2} b_{1n}\right], \quad (4.1.5)$$

$$s_2(\xi) = \frac{3}{8\,\xi^2}\sum_{n=-3}^{\infty}\left(-\frac{1}{2\,\xi}\right)^{n+2} b_{2n}, \quad (4.1.6)$$

where

$$\begin{aligned} b_{0n} &= \frac{1}{4}\left[n + 2 + \frac{2}{n+3} + \frac{8}{n+2} - \frac{16}{n+1}\right], \\ b_{1n} &= \frac{(n-1)\,(n+4)}{12} - \frac{2}{n+1}, \\ b_{2n} &= \frac{n^3 + 3\,n^2 - 2\,n - 24}{48}. \end{aligned} \quad (4.1.7)$$

The expansions for the functions S_j can be obtained from Eqs.(4.1.4) — (4.1.6).

4.2 Functions $\Psi_j(x,\gamma)$

The expansions for the functions $\psi_{ij}(\xi)$ and $\Psi_{ij}(\xi)$ in series for $\xi < 1/2$ have the same coefficients as (4.1.1). If we substitute the Eqs.(4.1.1) for the definitions (3.3.2) and (3.1.8), we obtain

$$\psi_{ij}(\xi) = \sum_{n=0}^{\infty} a_{jn}\,(-2\,\xi)^n\,\frac{i+1}{i+n+1}, \quad (4.2.1)$$

$$\psi_{-1j}(\xi) = \sum_{n=0}^{\infty} a_{j\,n+1}\,(-2\,\xi)^n\,\frac{-2}{n+1}, \quad (4.2.2)$$

$$\Psi_{ij}(\xi) = \sum_{n=0}^{\infty} A_{jn}\,(-2\,\xi)^n\,\frac{i+1}{i+n+1}. \quad (4.2.3)$$

Using (3.3.3) and (4.2.3), the expansions of $\Psi_j(x,\gamma)$ for $x\gamma < 1$ in a Taylor series can be deduced. For Ψ_0 we write the whole series:

$$\Psi_0(x,\gamma) = \sum_{n=0}^{\infty} a_{0n}\,(-2\,x)^n\,(n+1)!\sum_{k=0}^{\mathrm{Int}((n+1)/2)}\frac{\gamma^{n-2\,k}\,z^{2\,k}}{(2\,k+1)!\,(n+1-2\,k)!}. \quad (4.2.4)$$

Here $\mathrm{Int}(x)$ is the integral part of x. The expression correct up to three terms is the following

$$\Psi_0(x,\gamma) \sim 1 - \frac{2}{3}\,(4\,\gamma^2-1)\,\frac{x}{\gamma} + \frac{26}{5}\,(2\,\gamma^2-1)\,x^2. \tag{4.2.5}$$

For Ψ_1 and Ψ_2 we limit ourselves to the first three terms:

$$\begin{aligned}
\Psi_1(x,\gamma) &\sim \frac{1}{3}\,(4\,\gamma^2-1) - \frac{1}{5}\,(42\gamma^4 - 29\,\gamma^2+2)\,\frac{x}{\gamma} \\
&+ \frac{1}{25}\,(1176\,\gamma^4 - 1147\,\gamma^2 + 206)\,x^2, \\
\Psi_2(x,\gamma) &\sim \frac{1}{15}\,(42\,\gamma^4 - 34\,\gamma^2 + 7) - \frac{4}{75}\,(528\gamma^6 - 618\,\gamma^4 + 172\,\gamma^2 - 7)\,\frac{x}{\gamma} \\
&+ \frac{1}{525}\,(109120\,\gamma^6 - 158856\,\gamma^4 + 63677\,\gamma^2 - 6066)\,x^2.
\end{aligned} \tag{4.2.6}$$

The expansions found agree with the limiting values of these functions:

$$\Psi_j(x,1) = s_j(x). \tag{4.2.7}$$

In the same manner, using the expansions of s and S for large ξ, one can obtain the series for the functions ψ and Ψ for $x > \gamma$. Let us write them up to terms of order $1/x^7$:

$$\begin{aligned}
\Psi_0(x,\gamma) &\sim \frac{3}{8\,\gamma\,x^2}\left[\left(x - 2\,L_z - \frac{2}{x}\right)\,\ln 2\,x + \left(L_z\,\gamma - \frac{1}{2}\right)\,x + \frac{9}{2}\,L_z - \frac{7}{12}\frac{\gamma}{x^2}\right. \\
&\left. + \left(L_z\,\gamma - \frac{13}{8}\right)\,\frac{2}{x} + \frac{11}{576}\,\frac{4\,\gamma^2-1}{x^3} + \frac{\gamma}{60}\,\frac{2\,\gamma^2-1}{x^4} - \frac{11}{2400}\,\frac{16\,\gamma^4 - 12\,\gamma^2+1}{x^5}\right], \\
\Psi_1(x,\gamma) &\sim \frac{3}{8\,\gamma\,x^2}\left[\left(\gamma - L_z + \frac{3-2\,L_z\,\gamma}{x}\right)\,\ln 2\,x + \frac{4}{3}\,x + L_z\,\left(\gamma^2 - \frac{2}{3}\right) - \frac{11}{6}\,\gamma\right. \\
&+ \left(3\,L_z\,\gamma - \frac{13}{4}\right)\,\frac{1}{x} - \left(\frac{3}{2}\,L_z - \frac{11}{12}\right)\,\frac{1}{x^2} + \frac{2}{9}\,\frac{\gamma^2 - 25/16}{x^3} - \frac{29\,\gamma^2 - 32}{288}\,\frac{\gamma}{x^4} \\
&\left. + \frac{142\,\gamma^4 - 99\,\gamma^2 + 7}{2400}\,\frac{1}{x^5}\right].
\end{aligned} \tag{4.2.8}$$

$$\begin{aligned}
\Psi_2(x,\gamma) &\sim \frac{3}{8\,\gamma\,x^2}\left[\left(\frac{\gamma^2+2-3\,L_z\,\gamma}{x} - 2\frac{L_z\,(\gamma^2+1) - 2\,\gamma}{x^2}\right)\,\ln 2\,x + \frac{3}{4}\,x + \frac{7}{4}\,\gamma\right. \\
&- \frac{25}{12}\,L_z + \left(\gamma\,L_z\frac{12\,\gamma^2-17}{12} - \frac{58\,\gamma^2 - 73}{24}\right)\,\frac{1}{x} + \left(L_z\,\frac{76\,\gamma^2+25}{24} - \frac{13}{3}\gamma\right)\,\frac{1}{x^2} \\
&\left. + \left(2\,L_z\,\gamma - \frac{340\,\gamma^2 + 779}{576}\right)\,\frac{1}{x^3} + \frac{29\,\gamma^2 - 32}{288}\,\frac{\gamma}{x^4} - \frac{20\,\gamma^4 + 12\,\gamma^2 - 23}{576}\,\frac{1}{x^5}\right].
\end{aligned}$$

Here for short we use the notation $L_z = \ln{(\gamma+z)}/z$.

When we derive the asymptotics for $x \ll 1/\gamma$, all functions ψ and Ψ in formulae (3.3.3) are expanded in series on small xu, and for $x \gg \gamma$ the series on large xu are used. There is one more wide asymptotic region, when $x\gamma \gg 1$ and $\gamma \gg 1$, so $\gamma \gg \max\{1, 1/x\}$.

Then we may use the expansions of functions ψ and Ψ for large values of their arguments on the upper limit $u = \gamma + z \sim 2\gamma \gg 1$ only. Their values on the lower limit are not essential. As a result, for this region we have

$$
\begin{aligned}
\Psi_0(x,\gamma) &= \frac{3}{8}\,\frac{1}{x\,\gamma}\left(\ln 4x\gamma - \frac{1}{2}\right), \\
\Psi_1(x,\gamma) &= \frac{3}{8}\,\frac{1}{x^2}\left(\ln 4x\gamma - \frac{11}{6} + \frac{4}{3}\,\frac{x}{\gamma}\right), \\
\Psi_2(x,\gamma) &= \frac{3}{8}\,\frac{\gamma}{x^3}\,\left(\ln 4x\gamma - \frac{29}{12} + \frac{7}{4}\,\frac{x}{\gamma} + \frac{3}{4}\,\frac{x^2}{\gamma^2}\right).
\end{aligned}
\tag{4.2.9}
$$

All these formulae may be obtained from the asymptotics of corresponding functions (4.2.8) if we put $x \sim \gamma \gg 1$ in the leading terms. They hold as well when $\gamma \gg x$ (then ratios x/γ must be omitted) or when $x \gg \gamma \gg 1$ (then we must put $x/\gamma \to \infty$, but for this region we have more precise formulae). This is true of all formulae in such cases. For short we shall call this region $x \sim \gamma \gg 1$.

Let us now obtain the expansions of the quantities averaged over the Maxwellian distribution.

4.3 Maxwellian mean values

In order to deduce their asymptotic expansions, it is necessary to substitute the formulae like (4.1.1) for the definitions of the corresponding quantities. For $x < \min\{1, y\}$ we obtain

$$
\overline{s}_0(x) = \sum_{n=0}^{\infty}(-2\,x)^n\, a_{0n}\, f_{n+2}, \tag{4.3.1}
$$

$$
\overline{x_1}\,\overline{s}_0(x)/x = \sum_{n=0}^{\infty}(-2\,x)^n\,[a_{1n}\, f_{n+2} + (n+4)\, A_{1n}\, f_{n+3}/y], \tag{4.3.2}
$$

$$
\begin{aligned}
\overline{x_1^2}\,\overline{s}_0(x)/x^2 &= \sum_{n=0}^{\infty}(-2\,x)^n\,[a_{2n}\, f_{n+2} + 2\,(n+5)\, A_{3n}\, f_{n+3}/y \\
&+ (n^2 + 11\,n + 30)\, A_{4n}\, f_{n+4}/y^2],
\end{aligned}
\tag{4.3.3}
$$

Here we denote the ratio of the Macdonald functions

$$
f_n = f_n(y) = K_n(y)/K_2(y), \tag{4.3.4}
$$

and the argument y is omited for short. The expansions (4.3.1) and (4.3.2) are given in works by Shestakov et al. (1988) and Prasad et al. (1988).

For the asymptotic region $x > \max\{1, 1/y\}$, in the same manner one can obtain the expansions

$$
\overline{s}_0(x) = \frac{3}{8\,x^2}\left\{\left[\left(x - \frac{2}{x}\right)\, f_1 - 2\, f_0\right]\ln 2\,x + \left(\frac{x}{y} + \frac{9}{2} + \frac{2}{x\,y}\right)\, f_0\right.
$$

$$
\begin{aligned}
&+ \left(\frac{x}{2}-\frac{5}{4\,x}\right)\, f_1 + \sum_{n=0}^{\infty}\left(-\frac{1}{2\,x}\right)^{n+2} b_{0n}\, f_{n+2}\Bigg\}\,, \qquad (4.3.5)\\
\overline{x_1}\;\overline{s}_0(x) &= \frac{3}{8\,x}\left\{\left(\frac{4}{3}\,x-\frac{2}{3\,y}-\frac{1}{4\,x}-\frac{3}{y\,x^2}\right)\, f_1\right.\\
&+ \left[\left(\frac{2}{y}+\frac{1}{x}\right)\, f_1-\frac{2}{x\,y}\, f_0\right]\ln 2\,x+\left(\frac{2}{y^2}-\frac{3}{2}+\frac{3}{x\,y}\right)\, f_0\\
&- \left.\sum_{n=0}^{\infty}\left(-\frac{1}{2\,x}\right)^{n+2}\left[b_{1n}\, f_{n+2}-2\frac{n}{y}(b_{0\,n-1}+b_{1\,n-1})\, f_{n+1}\right]\right\}, \qquad (4.3.6)\\
\overline{x_1^2}\overline{s}_0(x)\,\frac{8}{3} &= \sum_{n=-3}^{\infty} b_{2n}\left(-\frac{1}{2\,x}\right)^{n+2} f_{n+2}+\frac{2}{y}\left[\left(1+3\,\frac{x}{y}\right)\, f_1\right.\\
&- \left.\frac{2}{y}\, f_0\right]\frac{\ln 2\,x}{x^2}+\left[\frac{7}{2}\left(1-\frac{1}{x\,y}\right)+\frac{1}{x\,y}\left(\frac{13}{12}-\frac{4}{x\,y}\right)\right]\frac{f_1}{y}\\
&+ \left[\frac{6}{y^2}-\frac{1}{3}\left(14-\frac{19}{x\,y}\right)\right]\frac{f_0}{x\,y}+\frac{1}{xy}\sum_{n=1}^{\infty}\left(-\frac{1}{2\,x}\right)^{n+2} n\, f_{n+2}\\
&\times \left[2\,(b_{1n}+b_{2n})+\frac{n+1}{x\,y}(b_{0n}+b_{2n}+2\,b_{1n})\right]. \qquad (4.3.7)
\end{aligned}
$$

The first terms in (4.3.1), (4.3.2), (4.3.5) and (4.3.6) were obtained earlier and are given, for example, in the papers by Guilbert (1986), Haruthyunian et al. (1988), Pozdnyakov et al. (1983).

The series (4.3.5) — (4.3.7) diverge and their summation is necessary, as usually for the asymptotic expansions, up to the term which is greater than previous one. Note that for large x the main contribution to the asymptotic expansions (4.3.5) — (4.3.7) is brought not by the series, but by the terms, which are out of the sign of summation.

As for functions $\Psi_j(x,\gamma)$, the asymptotics may be derived for Maxwellian mean functions in the region where $x \sim 1/y \gg 1$ or (more accurately) $y \ll \min\{x,1\}$. We can obtain them immediately from expressions (3.4.2) using the expansions of functions s_j and S_j such as (4.1.4) — (4.1.6) or from the leading terms of asymptotics (4.3.5) — (4.3.7). The result is the following

$$
\overline{s}_0(x) \sim \frac{3y}{16\,x}\left(\ln\frac{4x}{y}-\gamma_{_E}+\frac{1}{2}\right), \qquad (4.3.8)
$$

$$
\frac{\overline{x_1}}{x}\overline{s}_0(x) \sim \frac{3}{8\,x^2}\left(\ln\frac{4x}{y}-\gamma_{_E}-\frac{1}{3}+\frac{2}{3}xy\right), \qquad (4.3.9)
$$

$$
\frac{\overline{x_1^2}}{x^2}\overline{s}_0(x) \sim \frac{9}{8\,x^3\,y}\left(\ln\frac{4x}{y}-\gamma_{_E}-\frac{7}{12}\,(1-x\,y)+\frac{1}{8}x^2y^2\right). \qquad (4.3.10)
$$

Here $\gamma_{_E} = 0.577216$ is known as Euler's constant.

It is easy to verify that for $y \to \infty$ the quantities (3.4.2) tend to the correspondent functions $s_j(x)$. The next two terms of the expansion of these quantities in $1/y$ for large y

(i.e. electron distribution close to the non-relativistic Maxwellian) can be easily obtained. The expansions mentioned above have the form

$$\overline{x_1^j}\,\overline{s}_0(x)/x^j \sim s_j(x) + s_{j1}(x)/y + s_{j2}(x)/y^2. \tag{4.3.11}$$

The six functions which define the correction terms for $j = 0, 1$ and 2 are the following

$$s_{01}(x) = \frac{3}{16}\left[18\,x\,R_x^2 + 6\,R_x^4 - \left(3 - \frac{8}{x} - \frac{6}{x^2}\right)\frac{l_x}{x} - \frac{4}{x} - \frac{12}{x^2}\right], \tag{4.3.12}$$

$$\begin{aligned} s_{02}(x) &= \frac{3}{32}\left[\frac{3}{x}\left(\frac{5}{2} - \frac{8}{x} - \frac{5}{x^2}\right) l_x + \frac{18}{x} + \frac{30}{x^2} - 120\,x\,R_x^6 - 33\,R_x^4\right. \\ &+ \left. 18\,R_x^3 + \frac{73}{2}\,R_x^2 - \frac{89}{2}\,R_x\right], \end{aligned} \tag{4.3.13}$$

$$\begin{aligned} s_{11}(x) &= \frac{1}{16}\left[\frac{3}{x^3}\left(4 - \frac{7}{x}\right) l_x - \frac{66}{x} + \frac{42}{x^2} + 24\,R_x^5 + 18\,R_x^4\right. \\ &- \left. 2\,R_x^3 - 4\,R_x^2 + 108\,R_x\right], \end{aligned} \tag{4.3.14}$$

$$\begin{aligned} s_{12}(x) &= \frac{3}{64}\left[\frac{142}{x} - \frac{94}{x^2} - \left(24 - \frac{47}{x}\right)\frac{l_x}{x} + 240\,R_x^7 - 72\,R_x^5\right. \\ &- \left. 62\,R_x^4 + 94\,R_x^3 + 84\,R_x^2 - 244\,R_x\right], \end{aligned} \tag{4.3.15}$$

$$\begin{aligned} s_{21}(x) &= \frac{3}{4\,x^4}\left(l_x - \frac{8}{3}\,x^3 + 2\,x^2 - 2\,x\right) + \frac{1}{32}\left(60\,R_x^6 + 72\,R_x^5 + 51\,R_x^4\right. \\ &+ \left. 23\,R_x^3 + 109\,R_x^2 + 101\,R_x\right), \end{aligned} \tag{4.3.16}$$

$$\begin{aligned} s_{22}(x) &= \frac{3}{8}\left(6 - \frac{7}{x}\right)\frac{l_x}{x^3} + \frac{1}{4\,x}\left(46 - \frac{39}{x} - \frac{21}{x^2}\right) + \frac{1}{128}\left(2520\,R_x^8 + 1800\,R_x^7\right. \\ &+ \left. 1020\,R_x^6 + 240\,R_x^5 + 921\,R_x^4 + 1789\,R_x^3 + 983\,R_x^2 - 2809\,R_x\right). \end{aligned} \tag{4.3.17}$$

Formula (4.3.11) is valid for all x by virtue of the boundedness of the functions s_{ij}.

The expansion of the cross-section $\overline{s}_0(x)$ was obtained by Pozdnyakov et al. (1983), where the coefficients s_{01} and s_{02} were expressed through the derivatives of $x\,s_0(x)$ to the eighth order, inclusive. In the paper by Prasad et al. (1988) the whole expansion on $1/y$ was written, and its coefficients were defined by the differential recurrent relation.

Using the introduced functions, we shall study physical quantities in the following sections.

5 Energy balance, frequency dispersion and radiation pressure

5.1 Energy balance

As far as we are interested in the behaviour of the photon and electron gases, which does not depend on their marcoscopic motion, we shall consider a frame of references E and express all the quantities in the system of units $\hbar = c = m = 1$.

Let us consider the difference $x - x_1$, where x is the photon frequency before the scattering, x_1 is the frequency after the scattering defined by formula (2.3.6). This difference represents the energy which the photon of momentum $x\,\boldsymbol{\omega}$ loses because of the scattering by the electron of momentum $z\,\boldsymbol{\Omega}$, if the final photon moves in the direction $\boldsymbol{\omega}_1$. If this difference is negative, then $x_1 - x$ is the energy which the photon gains. The quantity $x- < x_1 >$ is the same energy averaged over the direction of the scattered photon. The product

$$(x- < x_1 >)\, n_\mathrm{e}\, \sigma_{_\mathrm{T}}\, s_0(\xi) = n_\mathrm{e}\, \sigma_{_\mathrm{T}}\, (x + x\,\xi - \gamma\,\xi)\, S_1(\xi) \tag{5.1.1}$$

is the energy which in average the photon of momenum $x\,\boldsymbol{\omega}$ loses as a result of the scattering by such electrons on the unit length. Further, the mean energy which the photon of frequency x loses in the process of scattering on the isotropic ensemble of electrons with a fixed energy is $x\,[1 - \Psi_1(x,\gamma)/\Psi_0(x,\gamma)]$. The product $x\,[\Psi_0(x,\gamma) - \Psi_1(x,\gamma)]\, n_\mathrm{e}\, \sigma_{_\mathrm{T}}$ is the energy loss of photon on the unit length at the layer of such electrons. Finally, $x - \overline{x_1}$ is the energy that the photon of frequency x loses in the thermal electron gas with a relativistic (in the general case) Maxwellian distribution function. Multiplying this quantity by $n_\mathrm{e}\, \sigma_{_\mathrm{T}}\, \overline{s}_0(x)$, we obtain the energy loss on the unit length of photon propagation. Obviously, this energy is transmited from photons to electrons and vice versa.

The energy balance between photons and electrons was considered by Barbosa (1982); Pozdnyakov et al. (1983); Guilbert (1986); Haruthyunian et al. (1988); Prasad et al. (1988); Shestakov et al. (1988), Coppi & Blandford (1990). Expression (5.1.1) was obtained by Haruthyunian et al. (1988) and Barbosa (1982). We supplement their results with those obtained by Nagirner et al. (1991).

Let us consider, first, the asymptotic expressions for the photon energy losses due to the scattering by the monoenergetic isotropic electron gas. For small frequencies of initial photon ($x \ll 1/\gamma$) the losses can be obtained using formulae (4.2.4) and (4.2.6). Thus, the losses on the unit length

$$\begin{aligned}\Psi_1(x,\gamma) - \Psi_0(x,\gamma) \;&\sim\; \frac{4}{3}\,z^2 - \frac{1}{15}\,(126\,\gamma^4 - 127\,\gamma^2 + 16)\frac{x}{\gamma} \\ &+\; \frac{1}{25}\,(1176\,\gamma^4 - 1407\,\gamma^2 + 336)\,x^2,\end{aligned} \tag{5.1.2}$$

and in respect to one scattering act

$$\begin{aligned}\Psi_1(x,\gamma)/\Psi_0(x,\gamma) - 1 \sim \frac{1}{3}\Bigg[4\,z^2 - \frac{1}{15}\,(218\,\gamma^4 - 181\,\gamma^2 + 8)\frac{x}{\gamma} \\ +\; \frac{1}{225}\,(13672\,\gamma^6 - 14529\,\gamma^4 + 2262\,\gamma^2 + 80)\frac{x^2}{\gamma^2}\Bigg].\end{aligned} \tag{5.1.3}$$

From expressions (5.1.2) and (5.1.3) it follows that in the case of immovable electrons and $x \ll 1$

$$\begin{aligned}\Psi_1(x,1) - \Psi_0(x,1) \;&\sim\; -x + \frac{21}{5}\,x^2, \\ \Psi_1(x,1)/\Psi_0(x,1) \;&\sim\; 1 - x + \frac{11}{5}\,x^2.\end{aligned} \tag{5.1.4}$$

In the opposite limiting case $x \gg \gamma$ we write only main terms of the expansion, though formulae (4.2.8) give a possibility of obtaining more detailed results:

$$\begin{aligned} \Psi_1(x,\gamma) - \Psi_0(x,\gamma) &\sim \frac{3}{8\,\gamma\,x^2}\left[-x\left(\ln 2\,x + L_z\,\gamma - \frac{11}{6}\right)\right. \\ &+ \left.(\gamma + L_z)\,\ln 2\,x + L_z\,\left(\gamma^2 - \frac{31}{6}\right) - \frac{11}{6}\gamma\right], \end{aligned} \tag{5.1.5}$$

$$\begin{aligned} &\Psi_1(x,\gamma)/\Psi_0(x,\gamma) - 1 \sim \\ &- \frac{\ln 2\,x + L_z\gamma - 11/6 - [(\gamma + L_z)\,\ln 2\,x + L_z\,(\gamma^2 - 31/6) - 11\gamma/6]/x}{\ln 2\,x + L_z\,\gamma - 1/2 - L_z\,(2\,\ln 2\,x - 9/2)/x}. \end{aligned} \tag{5.1.6}$$

The main terms in (5.1.5) and (5.1.6) (in the numerator and denominator) are given in the works by Pozdnyakov et al. (1983) and Guilbert (1986).

For the third asymptotic region $x \sim \gamma \gg 1$ from (4.2.9), we have

$$\frac{\Psi_1(x,\gamma)}{\Psi_0(x,\gamma)} \sim \frac{\gamma}{x}\left(1 - \frac{4}{3}\,\frac{1 - x/\gamma}{\ln 4x\gamma - 1/2}\right). \tag{5.1.7}$$

The similar asymptotic expressions are valid for the photon energy losses due to the scattering on the Maxwellian electrons. We write down these expressions, as above, in the same sequence and with the small number of the expansion terms. The results of the previous section permit us to move further, but the formulae become too cumbersome.

For $x \ll \min\{1, y\}$, using the notations (4.3.4), we obtain

$$\begin{aligned} (\overline{x_1} - x)\,\overline{s}_0(x)/x &\sim \frac{4}{y}\,f_3 - f_{34}x + \frac{21}{5}\,f_{45}\,x^2 \\ &\sim \begin{cases} 4/y - x, & x \gg 1/y^2, \\ 16\,(1 - 63\,x/2\,y)/y^2, & y^2 \ll x \ll y \ll 1, \end{cases} \end{aligned} \tag{5.1.8}$$

$$\begin{aligned} \overline{x_1}/x - 1 &\sim \frac{4}{y}f_3 - \left(f_{34} - \frac{8}{y}f_3^2\right)x + \left[\frac{21}{5}f_{45} + f_3\left(\frac{16}{y}f_3^2 - 2f_3 - \frac{314}{5y}f_4\right)\right]x^2 \\ &\sim \begin{cases} 4/y - x, & x \gg 1/y^2, \\ 16\,(1 - 47\,x/2\,y)/y^2, & y^2 \ll x \ll y \ll 1. \end{cases} \end{aligned} \tag{5.1.9}$$

Here is denoted $f_{k\,k+1} = f_k + 21\,f_{k+1}/y$, the first and the second lines marked with curly brackets correspond to the non-relativistic and to the ultrarelativistic limits, respectively.

For large frequencies, from (4.3.5) and (4.3.6) one can obtain

$$\begin{aligned} (x - \overline{x_1})\,\overline{s}_0(x) &\sim \frac{3}{8}\,f_1\left\{\ln 2\,x + \frac{f_*}{y} - \frac{5}{6} + \frac{1}{x}\left[-2\left(f_* + \frac{1}{y}\right)\,\ln 2\,x\right.\right. \\ &+ \left.\left.\frac{2}{3\,y} + \left(6 - \frac{2}{y^2}\right)\,f_*\right]\right\} \end{aligned}$$

$$
\sim \begin{cases} \dfrac{3}{8}\left(\ln 2\,x - \dfrac{5}{6}\right) - \dfrac{3}{4\,x}\,(\ln 2x - 3)\,, & 1 \ll x \ll y, \\ \dfrac{3}{16}\,y\left[\ln\dfrac{4\,x}{y} - \dfrac{5}{6} - \dfrac{2}{x\,y}\left(\ln\dfrac{4\,x}{y} - \dfrac{1}{3}\right)\right], & 1/\sqrt{x} \ll y \ll 1, \end{cases} \tag{5.1.10}
$$

where $f_* = f_0/f_1 = K_0(y)/K_1(y)$. To find $x - \overline{x_1}$, one must divide (5.1.10) by $\overline{s}_0(x)$. The expansion of this quantity for large frequencies with the same accuracy can be obtained from (4.3.5):

$$
\begin{aligned}
\overline{s}_0(x) &= 3\,f_1\,Z/8\,x,\ Z \sim \ln 2\,x + 1/2 + f_*/y - f_*\,(2\,\ln 2\,x - 9/2)\,/x \\
&\sim \begin{cases} \ln 2\,x + 1/2 - (2\,\ln 2\,x - 9/2)\,/x, & 1 \ll x \ll y, \\ \ln\dfrac{4\,x}{y} + \dfrac{1}{2} - \dfrac{y}{x}\,(2\,\ln 2\,x - \dfrac{9}{2})\,\ln\dfrac{2}{y}, & 1/\sqrt{x} \ll y \ll 1. \end{cases}
\end{aligned} \tag{5.1.11}
$$

Formulae (5.1.9) for $x \gg 1/y^2$ and (5.1.10) in the general case were deduced earlier by Pozdnyakov et al. (1983) and Guilbert (1986).

If $x \sim 1/y \gg 1$, then as follows from (4.3.8) and (4.3.9)

$$
\frac{\overline{x_1}}{x} \sim \frac{2}{x\,y}\left[1 - \frac{1}{6}\,\frac{5 - 4\,x\,y}{\ln(4x/y) - \gamma_E + 1/2}\right]. \tag{5.1.12}
$$

Calculations of the quantities Ψ_1/Ψ_0 and $\overline{x_1}/x$ are presented in Fig. 1 in the form of a set of contours, where their logarithms are constant. The values are indicated on the contour lines. The first quantity in Fig. 1a represents the ratio between the scattered and initial photon frequencies for the case of scattering on the monoenergetic isotropic electron gas as a function of electron momentum $z = \sqrt{\gamma^2 - 1}$ and initial photon frequency $x = k/mc$. The vertical lines correspond to the limit $x\gamma \ll 1$, where $\Psi_1/\Psi_0 \sim 4\gamma^2/3$ for $\gamma \gg 1$ and $\Psi_1/\Psi_0 \sim 1 + 4z^2/3$ for $z \ll 1$. The horizontal contour lines reflect the limit $z \ll 1$, where the photon frequency does not change if $x \ll 1$ (Thomson limit, Eq. (5.1.4)). In the opposite case of high-energy photons ($x \gg \gamma \sim 1$)

$$
\frac{\Psi_1(x,\gamma)}{\Psi_0(x,\gamma)} \sim \frac{4/3}{\ln 2x - 1/2}. \tag{5.1.13}
$$

This formula can be obtained from Eq. (5.1.6). In the region $\gamma \gg x,\ x\gamma \gg 1$ the ratio $\Psi_1/\Psi_0 \sim \gamma/x$ (inclined lines). All these limits are the special cases of asymptotic formula (5.1.7).

For the Maxwellian electrons (Fig. 1b) vertical lines correspond to the asymptotic region $x \ll \min\{1, y\}$ (see formula (5.1.9)); horizontal lines correspond to the non-relativistic case ($y \gg 1$), where $\overline{x_1}/x \sim s_1(x)/s_0(x)$ and the asymptotes for $x \ll 1$ and $x \gg 1$ are the same as for the monoenergetic electrons. Inclined lines correspond to (5.1.12).

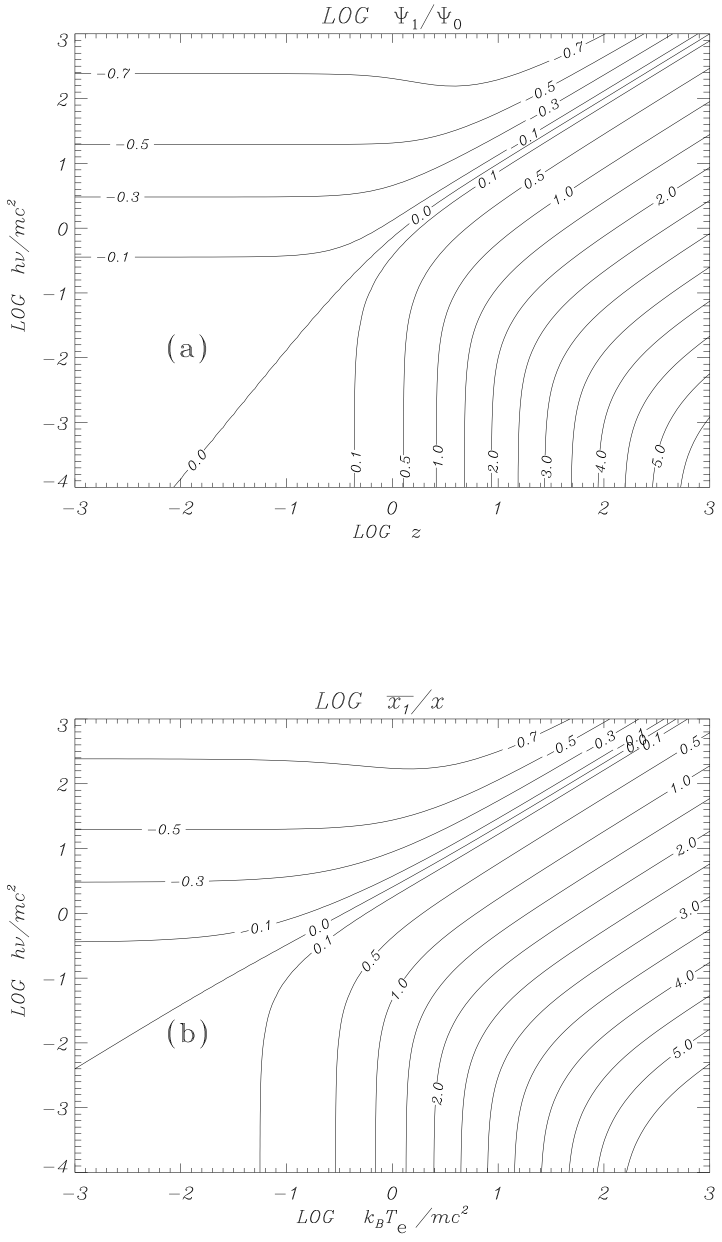

Figure 1: Mean frequency of the scattered photon in units of the initial frequency: (a) monoenergetic isotropic electron distribution; (b) Maxwellian electron gas.

5.2 Dispersion

The dispersion of the frequency of the scattered photon can be found with the usual formula, as a difference between the mean square and the square of mean of the frequency. Now we write down the asymptotic expressions for the same limiting cases as in the previous section. In the case of monoenergetic isotropic electrons the dispersion

$$D(x,\gamma) = x^2\left[\Psi_2(x,\gamma) - \Psi_1^2(x,\gamma)/\Psi_0(x,\gamma)\right]/\Psi_0(x,\gamma), \tag{5.2.1}$$

and for scattering by Maxwellian electrons the corresponding dispersion $\overline{D}(x,y) = \overline{x_1^2} - \overline{x_1}^2$.

Starting from the asymptotics of the mean powers of frequency, we obtain for small x, namely for $x \ll 1/\gamma$:

$$\begin{aligned} D(x,\gamma)/x^2 \sim (2/45)\,[(23\,\gamma^2 - 8)\,z^2 - x\,(2624\,\gamma^4 - 1120\,\gamma^2 - 19)\,z^2/15\,\gamma \\ + x^2\,(108826\,\gamma^8 - 174788\,\gamma^6 + 70047\,\gamma^4 - 3154\,\gamma^2 + 14)/105\,\gamma^2]. \end{aligned} \tag{5.2.2}$$

For large x

$$\begin{aligned} &D(x,\gamma)\left[\ln 2\,x + L_z\,\gamma - \frac{1}{2} - L_z\left(2\,\ln 2\,x - \frac{9}{2}\right)\frac{1}{x}\right]^2\frac{1}{x^2} \sim \\ &\sim \frac{3}{4}\,(\ln 2\,x + L_z\,\gamma) - \frac{155}{72} + \frac{1}{12\,x}\left[(14\,L_z - 11\,\gamma - 25)\,\ln 2\,x\right. \\ &\qquad \left. + L_z\left(\frac{371}{6} - 11\,\gamma^2 - 25\right) - \frac{1}{2}\,(21\,\gamma - 25)\right]. \end{aligned} \tag{5.2.3}$$

For the Maxwellian electrons and $x \ll \min\{1, y\}$

$$\begin{aligned} \overline{D}(x,y)\frac{1}{x^2} &\sim \frac{2}{y}\left(f_{34} - \frac{8}{y}\,f_3^2\right) - \frac{x}{y}\left[\frac{42}{5}\left(3\,f_4 + \frac{44}{y}\,f_5\right) + 4\,f_3\left(16\,f_3^2\right.\right. \\ &\left.\left. - 3\,f_{34}\right)\right] + x^2\left[\frac{7}{5}\left(1 - \frac{471}{y^2}\,f_4\right) - f_3^2\left(1 - \frac{40}{y}\,f_3 - \frac{192}{y^2}\,f_3^2 - \frac{5032}{5\,y}\,f_4\right)\right. \\ &\quad \left. + \frac{7}{5\,y}\,f_5\left(109 - \frac{1032}{y}\,f_3\right) - \frac{682}{5\,y}\left(f_3\,f_4 - \frac{16}{y}\,f_6\right)\right] \sim \\ &\sim \begin{cases} \dfrac{2}{y} - \dfrac{386}{5}\dfrac{x}{y} + \dfrac{2}{5}\,x^2, & x \gg 1/y, \\[2ex] \dfrac{752}{y^4} - \dfrac{233586}{5}\dfrac{x}{y^5}, & 1 \gg y \gg x \gg y^3. \end{cases} \end{aligned} \tag{5.2.4}$$

From formulae (5.2.2) and (5.2.4) it follows that in the case of cool electrons for small values of frequency x the dispersion is $D(x,1) = \overline{D}(x,\infty) \sim 2\,x^4/5$.

If the frequency is large, the Maxwellian dispersion

$$\overline{D}(x,y)\frac{Z^2}{x^2} \sim \left\{\frac{3}{4}\left(\ln 2\,x + \frac{f_*}{y}\right) - \frac{101}{72} - \frac{1}{x}\left[\frac{11}{6}\left(\left(\frac{1}{y} + f_*\right)\ln 2\,x + \frac{f_*}{y^2}\right)\right.\right.$$

$$
-\frac{127}{36\,y}-\frac{173}{24}f_*+\frac{1}{3}\frac{f_*^2}{y}\Bigg]\Bigg\}\sim \qquad (5.2.5)
$$

$$
\sim\begin{cases}\dfrac{3}{4}\ln 2\,x-\dfrac{101}{72}-\dfrac{1}{x}\left(\dfrac{11}{6}\ln 2\,x-\dfrac{173}{24}\right), & 1\ll x\ll y,\\[2ex] \dfrac{3}{4}\ln 2\,x-\dfrac{101}{72}-\dfrac{1}{x\,y}\left(\dfrac{11}{6}\ln\dfrac{4\,x}{y}-\dfrac{127}{36}\right), & 1/\sqrt{x}\ll y\ll 1.\end{cases}
$$

The logarithms of the quantities $D(x,\gamma)/x^2$ and $\overline{D}(x,y)/x^2$ are presented in Fig. 2a,b in the same manner as the mean frequency. The vertical lines in Fig. 2a correspond to the asymptotic region $x\gamma\ll 1$, where we can obtain the first term of the expansion from Eq.(5.2.2): $D/x^2\sim 2z^2/3$ if $x\ll z\ll 1$ and $D/x^2\sim 46\gamma^4/45$ if $\gamma\gg 1$. In the limit $z\ll x\ll 1$ dispersion $D/x^2\sim 2x^2/5$ and it is the same as in the limit $y\gg 1,\ x\ll 1$ for Maxwellian electron gas (Fig. 2b). These limits correspond to the horizontal lines. In the case $\gamma\sim x,\ x\gamma\gg 1$ (inclined lines)

$$
\frac{D(x,\gamma)}{x^2}\sim\frac{\gamma^2}{x^2}\,\frac{3\,(1-x/\gamma+x^2/\gamma^2)\,\ln 4x\gamma-(155-289x/\gamma+27x^2/\gamma^2)/18}{4[\ln 4x\gamma-1/2]^2}. \qquad (5.2.6)
$$

For the Maxwellian electrons the dispersion in the asymptotic region $x\ll\min\{1,y\}$ (formula 5.2.4) corresponds to the vertical lines and in the region $y\ll\min\{x,1\}$ the inclined lines reflect the formula

$$
\overline{D}(x,y)\sim\frac{2}{y^2}\left[1+\frac{1}{24}\,\frac{2-22xy+9x^2y^2}{\ln(4x/y)-\gamma_{_E}+1/2}-\frac{1}{18}\left(\frac{5-4xy}{\ln(4x/y)-\gamma_{_E}+1/2}\right)^2\right]. \qquad (5.2.7)
$$

In the next section we shall study the effects of the radiation pressure on the electron gas.

5.3 Radiation pressure

The last quantity we consider as an application of the developed formalism of the mean values of single Compton scattering is the radiation pressure on the electron gas. This problem was considered by Pozdnyakov et al. (1983), where the calculations by the Monte-Carlo method were performed as well.

It is easy to see that if the monodirectional (along the vector $\boldsymbol{\omega}$ in the frame E) photons interact with the electron gas, then, by virtue of the isotropic distribution of the electron momenta, these photons transmit to the gas (on average) the momentum in the same direction $\boldsymbol{\omega}$. The photon of momentum $x\,\boldsymbol{\omega}$ transmits to the electron of momentum $z\,\boldsymbol{\Omega}$ the momentum $x\,\boldsymbol{\omega}-x_1\,\boldsymbol{\omega}_1$, if after scattering it has the direction $\boldsymbol{\omega}_1$. The frequency x_1 is defined by Eq.(2.3.6). Therefore, the mean momentum which the scattered photons transmit in the direction $\boldsymbol{\omega}$ is

$$
<x-x_1\,\mu>\ s_0(\xi)=(x-<x_1>)\,s_0(\xi)+<x_1\,(1-\mu)>\ s_0(\xi). \qquad (5.3.1)
$$

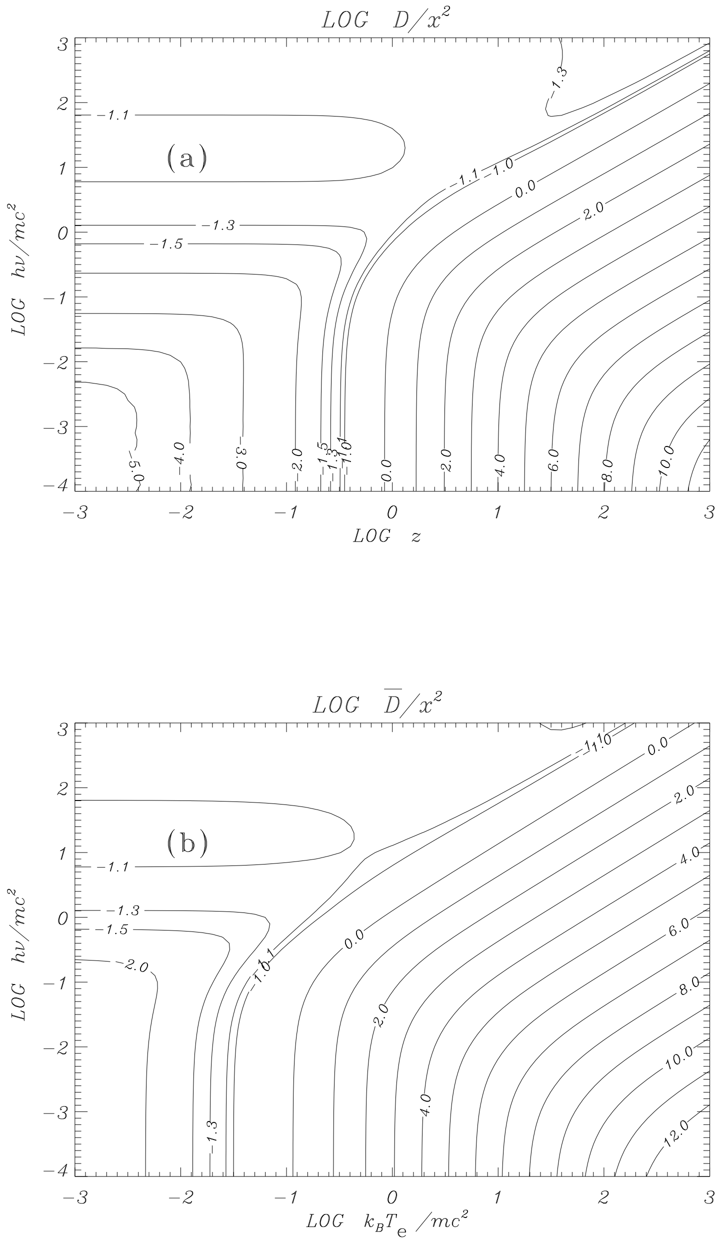

Figure 2: Dispersion of the frequency of the scattered photon: (a) scattering on the monoenergetic electrons; (b) scattering on the Maxwellian electrons.

The additional term by which the transmited momentum differs from the energy transmited, does not require the introduction of new functions. In fact, using Eqs. (2.3.11), (3.2.4), (3.2.6), (3.2.11) and (3.2.18), we obtain

$$
\begin{aligned}
< x_1(1-\mu) > s_0(\xi) = \frac{3}{16\pi\xi^2}\int x_1^3(1-\mu)F\mathrm{d}^2\omega_1 \\
= \frac{3}{16\,\pi\,x\,\xi}\int \xi_1\,(1-\mu_0)\,F\,\mathrm{d}\xi_1\,\mathrm{d}\phi_0 = \frac{3}{8\,x\,\xi^2}\int_{\xi/(1+2\,\xi)}^{\xi}(\xi-\xi_1)\,F\,\mathrm{d}\xi_1 \\
= \frac{\xi}{x}\,[s_0(\xi)-s_1(\xi)] = \frac{\xi^2}{x}\,S_1(\xi). \qquad (5.3.2)
\end{aligned}
$$

Thus, if we fix the electron and photon energies and the angle between their momenta, then the mean momentum transmited to the electron gas in the direction of the initial photons' propagation in accordance with (3.2.18), (3.2.16) and (5.3.2) is

$$
(x- < x_1\,\mu >)\,s_0(\xi) = (x + x\,\xi - \xi\,\gamma + \xi^2/x)\,S_1(\xi). \qquad (5.3.3)
$$

The averaging of this quantity over the directions of the electron momenta adds to $\Psi_0(x,\gamma)-\Psi_1(x,\gamma)$ the mean value of $\xi^2 S_1(\xi)/x$, which we denote as

$$
\begin{aligned}
\Psi_1^*(x,\gamma) &= \frac{1}{2\,\gamma\,z\,x^3}\int_{x\,(\gamma-z)}^{x\,(\gamma+z)} < x_1\,(1-\mu) > \,\xi\,s_0(\xi)\,\mathrm{d}\xi \\
&= \frac{1}{2\,\gamma\,z\,x^4}\int_{x\,(\gamma-z)}^{x\,(\gamma+z)} \xi^3\,S_1(\xi)\,\mathrm{d}\xi = \frac{1}{2\,\gamma\,z}\,\frac{u^4}{4}\,\Psi_{31}(x\,u)\Big|_{\gamma-z}^{\gamma+z}. \qquad (5.3.4)
\end{aligned}
$$

Further, the momentum which is transmited to the monoenergetic isotropic electron gas in respect of one photon and unit length we denote as $x\,\Psi^*(x,\gamma)\,n_{\mathrm{e}}\,\sigma_{\mathrm{T}}$. And here

$$
\Psi^*(x,\gamma) = \Psi_0(x,\gamma) - \Psi_1(x,\gamma) + \Psi_1^*(x,\gamma). \qquad (5.3.5)
$$

To write down the expression for the function $\Psi_{31}(\xi)$ which enters (5.3.4), it is necessary to introduce the twelfth function like (3.3.2):

$$
\psi_{20}(\xi) = \frac{9}{8\,\xi^3}\left[\frac{25}{4}\,\xi + \left(\frac{\xi^2}{2} - 2\,\xi - \frac{5}{4}\right)\,l_\xi + \frac{\xi}{4}\,R_\xi - 2\,g(\xi)\right]. \qquad (5.3.6)
$$

Then

$$
\Psi_{31}(\xi) = 4\,[\psi_{20}(\xi) - \psi_{21}(\xi)]/3\,\xi. \qquad (5.3.7)
$$

Averaging (5.3.4) over (2.1.8), we obtain the difference between the momentum transmited to the electron gas and the transmited energy, i.e.

$$
\begin{aligned}
\overline{x_1\,(1-\mu)}\,\overline{s}_0(x) &= 4\,\pi\,x\int_1^\infty z\,\gamma\,f_{\mathrm{M}}(\gamma)\,\mathrm{d}\gamma\,\Psi_1^*(x,\gamma) \\
&= \frac{x}{2\,K_2(y)}\int_1^\infty \exp\,(-y\,\gamma)\frac{\mathrm{d}\gamma}{z}u^4\,S_1(x\,u)\Big|_{\gamma-z}^{\gamma+z}(+). \qquad (5.3.8)
\end{aligned}
$$

The values of the transmited energies $\Psi_1(x,\gamma) - \Psi_0(x,\gamma)$ or $(\overline{x_1} - x)\,\overline{s}(x)$ can be positive or negative, but the mean transmited momentum is always positive.

Let us now present here the asymptotics of the functions defining the radiation pressure. First, we write down the formulae for the additional term, then for the force of radiation pressure, or, to be precise, for its dimensionless factor divided by x.

For $x < 1/\gamma$ one obtains the expansion similar to (4.2.4):

$$\Psi_1^*(x,\gamma) = \sum_{n=0}^{\infty}(-2\,x)^n (n+3)!A_{1n} \sum_{k=0}^{\mathrm{Int}((n+3)/2)} \frac{\gamma^{n+2-2k}z^{2k}}{(2k+1)!(n+3-2k)!}, \tag{5.3.9}$$

and for $x > \gamma$

$$\begin{aligned}\Psi_1^*(x,\gamma) &= \frac{3}{8\,\gamma\,x^2}\left[\left(\gamma - \frac{2}{x} - 3\frac{L_z}{x^2}\right)\ln 2\,x + \frac{2\,\gamma^2-1}{2}\,L_z - \frac{4}{3}\gamma - \frac{4+2\,L_z\,\gamma}{x} - \frac{L_z}{x^2}\right.\\ &- \left. 2\sum_{n=0}^{\infty}\left(-\frac{1}{2\,x}\right)^{n+3} n!(b_{0n}+b_{1n}) \sum_{k=0}^{\mathrm{Int}(n/2)} \frac{\gamma^{n-2k}z^{2k}}{(2k+1)!(n-2k)!}\right].\end{aligned} \tag{5.3.10}$$

Then the first terms of the expansion of the function $\Psi^*(x,\gamma)$ are the following, respectively, for $x \ll 1/\gamma$ and $x \gg \gamma$:

$$\begin{aligned}\Psi^*(x,\gamma) &\sim \frac{2\gamma^2+1}{3} - \frac{378\gamma^4 - 121\gamma^2 - 17}{75}\,\frac{x}{\gamma}\\ &+ \frac{4704\gamma^4 - 3318\gamma^2 + 189}{150}x^2,\end{aligned} \tag{5.3.11}$$

$$\Psi^*(x,\gamma) \sim \frac{3}{8\gamma x^2}\left[x\left(\ln 2\,x + L_z\gamma - \frac{11}{6}\right) - L_z\left(\ln 2\,x - \frac{14}{3}\right) + \frac{\gamma}{2}\right]. \tag{5.3.12}$$

For the third asymptotic region $x \sim \gamma \gg 1$ we obtain

$$\Psi_1^*(x,\gamma) \sim \frac{3}{8}\frac{1}{x^2}\left(\ln 4x\gamma - \frac{4}{3}\right),\quad \Psi^*(x,\gamma) \sim \frac{3}{8}\frac{1}{x^2}\left[\frac{1}{2} + \frac{x}{\gamma}\left(\ln 4x\gamma - \frac{11}{6}\right)\right]. \tag{5.3.13}$$

Operating in the same manner, we find that for the Maxwellian mean values the asymptotic expansions have the form for $x < \min\{1, y\}$:

$$\overline{x_1(1-\mu)\overline{s}_0(x)} = x\sum_{n=0}^{\infty}(-2x)^n A_{1n} f_{n+4}, \tag{5.3.14}$$

and for $x > \max\{1, 1/y\}$:

$$\begin{aligned}\overline{x_1(1-\mu)}\overline{s}_0(x) &= \frac{3}{8x^2}\left[\left(x - 2f_1 - \frac{3}{x}f_0\right)\ln 2\,x - \frac{5}{6}x + 6f_1 - \frac{1}{x}f_0\right.\\ &+ \left.\frac{2}{y}\left(xf_1 + x\frac{f_0}{y} - f_0\right) + \sum_{n=0}^{\infty}(b_{0n}+b_{1n})\left(-\frac{1}{2x}\right)^{n+2} f_{n+1}\right].\end{aligned} \tag{5.3.15}$$

The first two terms of the expansion of the mean transmited momentum for the same values of x are, respectively,

$$
\begin{aligned}
(x - \overline{x_1\,\mu})\overline{s}_0(x)/x &\sim 1 + \frac{2}{y}f_3 - \left(16 f_3 + \frac{63}{y}f_4\right)\frac{x}{5} \\
&\sim \begin{cases} 1 - \dfrac{16}{5}x, & 1/y \ll x \ll 1, \\[2ex] \dfrac{8}{y^2} - \dfrac{1512}{5}\dfrac{x}{y^3}, & y^3 \ll x \ll y \ll 1, \end{cases}
\end{aligned} \tag{5.3.16}
$$

$$
\begin{aligned}
(x - \overline{x_1\,\mu})\overline{s}_0(x) &\sim \frac{3}{8}f_1\left\{\ln 2\,x - \frac{5}{6} + \frac{f_*}{y} + \frac{1}{x}\left[f_*\left(\ln 2\,x - \frac{31}{6}\right) - \frac{1}{y}\right]\right\} \\
&\sim \begin{cases} \dfrac{3}{8}\left[\ln 2\,x - \dfrac{5}{6} - \dfrac{1}{x}\left(\ln 2\,x - \dfrac{31}{6}\right)\right], & 1 \ll x \ll y, \\[2ex] \dfrac{3}{16}y\left(\ln\dfrac{4x}{y} - \dfrac{5}{6} + \dfrac{1}{xy}\right), & 1/\sqrt{x} \ll y \ll 1. \end{cases}
\end{aligned} \tag{5.3.17}
$$

The notation are the same as above.

We note that for small frequencies x the pressure is larger than the energy losses in both ultrarelativistic and non-relativistic cases. For large initial photon energy both quantities are of the same order and differ only by terms of order $\sim 1/x$ (formulae 5.1.10 and 5.3.17). Note also that if in Eq. (5.3.17) for $1/\sqrt{x} \ll y \ll 1$ we omit the terms of order $1/x$, then the expression obtained will be valid for $y \ll 1/x \ll 1$, i.e. for the ultrarelativistic electrons, energies of which are larger than the photon energy. Formulae (5.3.16) for $1/y \ll x \ll 1$ and (5.3.17) for $1 \ll x \ll y$ are given in the paper by Pozdnyakov et al. (1983).

Finally, let us write down the expansion of the additional term (5.3.8) for $y \gg 1$, i.e. in the nearly non-relativistic case:

$$
\overline{x_1(1-\mu)}\;\overline{s}_0(x)/x \sim S_1(x) + s^*_{11}(x)/y + s^*_{12}(x)/y^2, \tag{5.3.18}
$$

where

$$
\begin{aligned}
s^*_{11}(x) &= \frac{3}{8\,x}\left[\frac{3}{x^2}\left(1 + \frac{2}{x}\right)\,l_x + \frac{6}{x} - \frac{12}{x^2} - 4\,R_x^5\right. \\
&+ \left. 2\,R_x^4 - 8\,x\,R_x^3 + 20\,R_x\right],
\end{aligned} \tag{5.3.19}
$$

$$
\begin{aligned}
s^*_{12}(x) &= \frac{3}{32\,x}\left[\frac{72}{x^2} - \frac{42}{x} - \frac{3}{x^2}\left(5 + \frac{12}{x}\right)\,l_x - 120\,R_x^7 + 140\,R_x^6 + 32\,R_x^5\right. \\
&- \left. 46\,R_x^4 - 48\,R_x^3 + 32\,R_x^2 + 76\,R_x\right].
\end{aligned} \tag{5.3.20}
$$

As in the previous two sections, we present the radiation pressure in contour form (Fig. 3). Thus, in the case of monoenergetic electrons (Fig. 3a) in the limit $x\gamma^3 \ll 1$, $\gamma \gg$

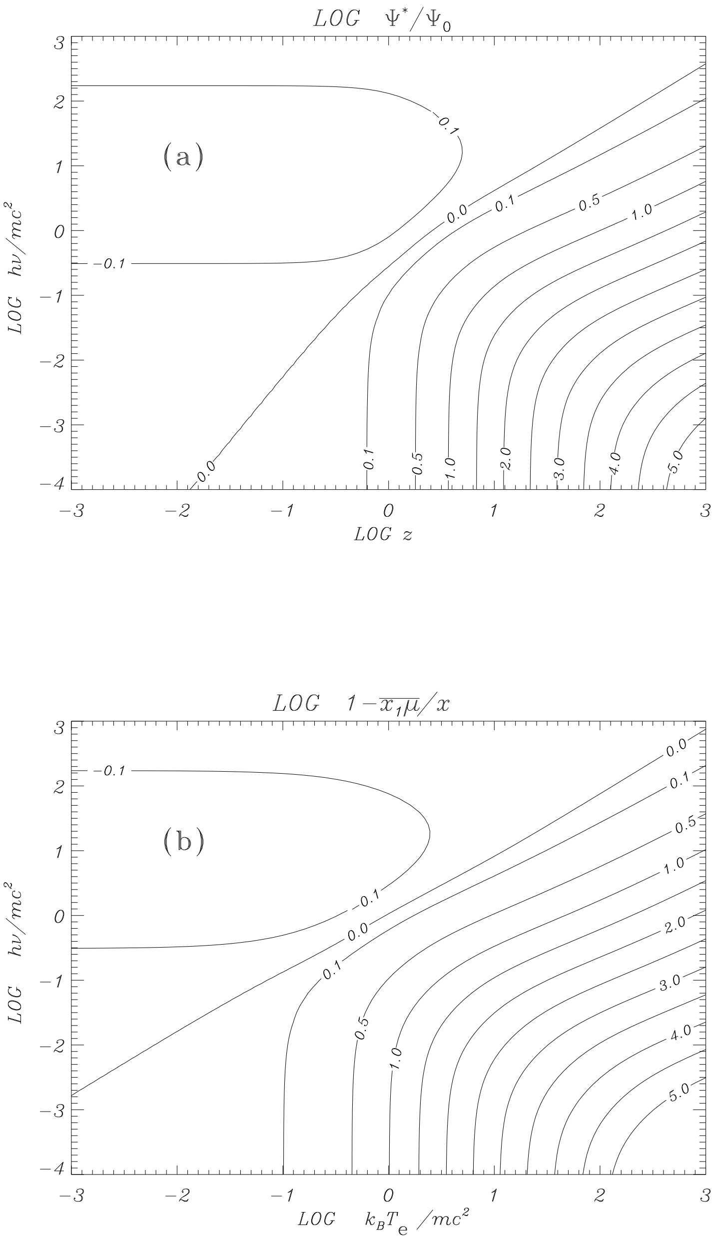

Figure 3: Radiation pressure in units of the scattering cross-section: (a) monoenergetic electrons; (b) Maxwellian electron distribution.

1 pressure $\Psi^*/\Psi_0 \sim 2\gamma^2/3$ (vertical lines) and in the asymptotic region $x\,\gamma \gg 1,\ \gamma \gg 1$ or $x \sim \gamma \gg 1$

$$\frac{\Psi^*(x,\gamma)}{\Psi_0(x,\gamma)} \sim 1 + \frac{\gamma/2x - 4/3}{\ln 4x\gamma - 1/2}\,, \tag{5.3.21}$$

which corresponds to the inclined lines.

For the Maxwellian electrons the vertical lines correspond to the limit $x \ll \min\{1, y\}$ (Eq. (5.3.16)) and the inclined lines are described by the formula

$$1 - \frac{\overline{x_1\,\mu}}{x} \sim 1 + \frac{1/3 + 1/xy}{\ln{(4x/y)} - \gamma_{_E} + 1/2}. \tag{5.3.22}$$

We have now finished the consideration of the results connected with the term of the kinetic equation describing the extinction of the radiation due to the non-induced Compton scattering. In the following sections we shall study the scattering matrix which defines the term describing properly the photon scattering.

6 Scattering matrix and redistribution functions

6.1 Rotation angles

The scattering matrix describing the scattering with the transformation of the photon $\mathbf{k}_1$ to the photon $\mathbf{k}$ was defined by Eq. (2.4.5). We considered the inverse scattering in the sections 2.2 and 2.3, i.e. the initial particles were the photon of momentum $\mathbf{k}$ and the electron of momentum $\mathbf{p}$. The expressions for the unit vectors of the *internal* polarization basis, for scattering matrix and for all other quantities, were written for that case. In the paper by Nagirner & Poutanen (1993b) matrix (2.4.5) was calculated in such a way that the initial and final electrons were replaced without changing photons. It is possible to do this by virtue of the specific properties of the *internal* polarization basis. Here we deduce the scattering matrix in another way, considering the scattering act $\mathbf{k}, \mathbf{p} \to \mathbf{k}_1, \mathbf{p}_1$. Let us rewrite definition (2.4.5) for that process

$$\hat{R}(\mathbf{k} \to \mathbf{k}_1) = \frac{3}{16\pi}\frac{m^3c^3}{n_\mathrm{e}}\int\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p_1}{p_{10}}\delta(\underline{p}_1 + \underline{k}_1 - \underline{p} - \underline{k}) f_\mathrm{e}(\mathbf{p})\hat{L}(-\chi_1)\hat{F}\hat{L}(\chi). \tag{6.1.1}$$

The triple integral over $\mathbf{p}_1$ vanishes, owing to the existence of the three-dimensional δ-function. Then we must substitute $\mathbf{p}_1 = \mathbf{p} + \mathbf{k} - \mathbf{k}_1$ into all quantities in the integrand. Using (2.3.2) and identity (3.1.2), we write down (6.1.1) in the form

$$\hat{R}(\mathbf{k} \to \mathbf{k}_1) = \frac{3}{16\pi}\frac{m^3c^3}{n_\mathrm{e}}\int\frac{\mathrm{d}^3p}{p_0}\delta(\underline{k}_1 \cdot (\underline{p} + \underline{k}) - \underline{k} \cdot \underline{p})\, f_\mathrm{e}(\mathbf{p})\, \hat{L}(-\chi_1)\, \hat{F}\, \hat{L}(\chi). \tag{6.1.2}$$

Now we pass to the frame E and take into account the isotropy of the electron distribution on momenta. Then the scattering matrix takes the form

$$\begin{aligned}\hat{R}(x, \boldsymbol{\omega} \to x_1, \boldsymbol{\omega}_1) &= \\ &= \frac{3}{16\pi}\int\frac{\mathrm{d}^3z}{\gamma}\delta(x_1\,(\gamma - z\,\zeta_1) + q - x\,(\gamma - z\,\zeta))\, f_\mathrm{e}(\gamma)\, \hat{L}(-\chi_1)\, \hat{F}\, \hat{L}(\chi).\end{aligned} \tag{6.1.3}$$

We have to calculate the triple integral from the matrix function. It contains of the Stokes parameters' transformation matrices, which depend on the angles between the vectors of the polarization bases. Therefore, it is necessary to find these angles.

Now we give the expressions for the cosines and sines of the rotation angles of the polarization bases in the frame E. First, for the rotation angle χ^0 between the *internal* basis (2.3.10) of the photon $\boldsymbol{\omega}$ and the *external* basis (2.2.8) for the scattering by the electron at rest:

$$\cos\chi^0 = (\eta_1 - \mu\,\eta)/\sqrt{1-\eta^2}\,\sqrt{1-\mu^2}, \quad \sin\chi^0 = -C_l/\sqrt{1-\eta^2}\,\sqrt{1-\mu^2}, \tag{6.1.4}$$

where $C_l = (\mathbf{l}\,\boldsymbol{\omega}\,\boldsymbol{\omega}_1)$. In the same manner for the photon $\boldsymbol{\omega}_1$

$$\cos\chi_1^0 = (\eta - \mu\,\eta_1)/\sqrt{1-\eta_1^2}\,\sqrt{1-\mu^2}, \quad \sin\chi_1^0 = C_l/\sqrt{1-\eta_1^2}\,\sqrt{1-\mu^2}, \tag{6.1.5}$$

which can be obtained from (6.1.4) by changing $\boldsymbol{\omega} \leftrightarrow \boldsymbol{\omega}_1$.

Then let us find the angles of rotation χ and χ_1 for the same photons, but for the scattering on the electron of the arbitrary momentum. Using Eqs. (2.2.3), (2.2.8) and (2.3.10) and some relations for the products of four-vectors (see Nagirner & Poutanen, 1993b for detail), we obtain for the initial photon

$$\begin{aligned}
\cos\chi &= \frac{\gamma\,(\eta_1 - \mu\,\eta) - z\,[w\,\kappa + (\eta_1 - \eta)\,\zeta]}{\sqrt{1-\eta^2}\,w\,\Delta}, \\
\sin\chi &= -\frac{(\gamma - z\,\zeta)\,C_l - z\,w\,C_\omega}{\sqrt{1-\eta^2}\,w\,\Delta}.
\end{aligned} \tag{6.1.6}$$

Here Δ is defined by (2.3.11) and

$$\begin{aligned}
C_\omega &= (\mathbf{l}\,\boldsymbol{\omega}\,\boldsymbol{\Omega}) = \frac{\sqrt{1-\zeta^2}}{\sqrt{1-\mu^2}}\,[C_l\,\cos\Phi - (\eta_1 - \mu\,\eta)\,\sin\Phi], &(6.1.7)\\
\kappa &= \boldsymbol{\Omega}\cdot\mathbf{l} = \eta\,\zeta + \frac{\sqrt{1-\zeta^2}}{\sqrt{1-\mu^2}}\,[(\eta_1 - \mu\,\eta)\,\cos\Phi + C_l\,\sin\Phi]. &(6.1.8)
\end{aligned}$$

We note that Φ is the azimuth of the vector $\boldsymbol{\Omega}$ in the basis (2.3.18), the quantity μ_0 was defined in (2.3.11) and μ is the cosine of the scattering angle.

For photon $\boldsymbol{\omega}_1$ the expressions can be obtained by changing $\boldsymbol{\omega} \leftrightarrow \boldsymbol{\omega}_1$:

$$\begin{aligned}
\cos\chi_1 &= \frac{\gamma\,(\eta - \mu\,\eta_1) - z\,[w\,\kappa + (\eta - \eta_1)\,\zeta_1]}{\sqrt{1-\eta_1^2}\,w\,\Delta}, \\
\sin\chi_1 &= \frac{(\gamma - z\,\zeta_1)\,C_l + z\,w\,C_{\omega_1}}{\sqrt{1-\eta_1^2}\,w\,\Delta},
\end{aligned} \tag{6.1.9}$$

where

$$C_{\omega_1} = (\mathbf{l}\,\boldsymbol{\omega}_1\,\boldsymbol{\Omega}) = -C_l\,\zeta + \frac{\sqrt{1-\zeta^2}}{\sqrt{1-\mu^2}}\,[\mu\,C_l\,\cos\Phi - (\eta - \mu\,\eta_1)\,\sin\Phi]\,. \tag{6.1.10}$$

Expressions (6.1.4) and (6.1.5) can be obtained from (6.1.6) and (6.1.9) for $z = 0$.

Substituting the expressions for κ, C_ω and C_{ω_1} into (6.1.6) and (6.1.9) and denoting

$$\begin{aligned} \cos\Psi &= \left[(\gamma - z\,\zeta)\,\sqrt{r} - z\,\sqrt{1-\zeta^2}\,\cos\Phi\right]/\Delta, && (6.1.11)\\ \sin\Psi &= z\,\sqrt{1-\zeta^2}\,\sin\Phi/\Delta, && (6.1.12) \end{aligned}$$

we find that in the frame E

$$\chi = \chi^0 - \Psi, \quad \chi_1 = \chi_1^0 + \Psi. \tag{6.1.13}$$

6.2 Calculation of integral over electron directions

Representing the triple integral in (6.1.3) as the sequence of three integrals over the separate variables and substituting instead of the scalar products in the argument of the δ-function their expressions through the introduced angles, we get

$$\begin{aligned} \hat{R}(x, \boldsymbol{\omega} \to x_1, \boldsymbol{\omega}_1) &= \frac{3}{16\pi}\int_0^\infty \frac{z^2 \mathrm{d}z}{\gamma} f_\mathrm{e}(\gamma)\int_{-1}^1 \mathrm{d}\zeta \int_0^{2\pi} \mathrm{d}\Phi \hat{L}(-\chi_1)\hat{F}\hat{L}(\chi) \\ &\delta\left(\gamma(x_1 - x) + q - z\,[x_1\,(\zeta\,\mu + \sqrt{1-\zeta^2}\sqrt{1-\mu^2}\cos\Phi) - x\zeta]\right). \end{aligned} \tag{6.2.1}$$

First, we calculate the integral over the azimuth Φ of the direction of the electron momentum. We do not write down the matrix, which is the product of the matrices in the integrand, but note that only cosines and sines of 2χ and $2\chi_1$ depend on Φ through the quantity Ψ according to Eq. (6.1.13). This quantity is defined by formulae (6.1.11) and (6.1.12).

Calculating the integrals over Φ, we observe that the argument of the δ-function becomes equal to zero if $\cos\Phi = \cos\Phi^0$, where

$$\cos\Phi^0 = \frac{x_1(\gamma - z\zeta\mu) - x(\gamma - z\zeta) + q}{x_1\,z\,\sqrt{1-\zeta^2}\sqrt{1-\mu^2}} = \frac{(x_1\mu - x)\xi + q(\gamma + x)}{q\,z\,\sqrt{1-\zeta^2}\,\sqrt{r}}, \tag{6.2.2}$$

and r is defined in (2.3.17).

The integrals are not zero if the absolute value of $\cos\Phi = \cos\Phi^0$ is not larger than 1, or

$$D_*^2 = z^2(1-\zeta^2)r - \frac{1}{x_1^2 w^2}[x_1(\gamma - z\zeta\mu) - x(\gamma - z\zeta) + q]^2 \geq 0. \tag{6.2.3}$$

To satisfy this condition, the values of ζ must be between the roots of the left hand side of inequality (6.2.3)

$$\zeta^\pm = \frac{[\gamma(x_1 - x) + q](x_1\mu - x) \pm x_1 w b_*}{zQ^2}, \tag{6.2.4}$$

where

$$b_* = \sqrt{r}\sqrt{z^2Q^2 - [\gamma(x_1 - x) + q]^2}, \quad Q^2 = (x - x_1)^2 + 2q. \tag{6.2.5}$$

For the existence of the real roots $\zeta^{\pm}$ of D_*^2, it is necessary that

$$z^2Q^2 - [\gamma(x_1 - x) + q]^2 \geq 0. \tag{6.2.6}$$

Under these conditions there are two values of the azimuth Φ in the interval $[0, 2\pi]$ where the argument of the δ-function becomes zero: $\Phi = \Phi^0 = \arccos\cos\Phi^0$ and $\Phi = 2\pi - \Phi^0$. The values of $\sin\Phi$ at these points differ by the sign $\sin\Phi = \pm\sin\Phi^0$, where

$$\sin\Phi^0 = \frac{D_*}{z\sqrt{1-\zeta^2}\sqrt{r}} \geq 0. \tag{6.2.7}$$

Owing to these facts the integrals over Φ from all quantities proportional to $\cos\Phi$ must be taken into account twice, but those proportional to $\sin\Phi$ enter with the opposite signs and vanish. The integral over Φ from the δ-function

$$\int_0^{2\pi} \delta\left(\gamma(x_1 - x) + q - z[x_1(\zeta\mu + \sqrt{1-\zeta^2}\sqrt{1-\mu^2}\cos\Phi) - x\zeta]\right) \mathrm{d}\Phi =$$
$$= \frac{2}{\sin\Phi^0\, z\, x_1\,\sqrt{1-\zeta^2}\,\sqrt{1-\mu^2}} = \frac{2x}{D_*\, q}. \tag{6.2.8}$$

After calculating integrals over Φ the structure of the matrix $\hat{L}(-\chi_1)\hat{F}\hat{L}(\chi)$ simplifies immensely. Actually,

$$\int_0^{2\pi} \delta\left(\gamma(x_1 - x) + q - z\left[x_1(\zeta\mu + \sqrt{1-\zeta^2}\sqrt{1-\mu^2}\cos\Phi) - x\zeta\right]\right)$$
$$\times\, \hat{L}(-\chi_1)\hat{F}\hat{L}(\chi)\mathrm{d}\Phi = \frac{2x}{D_*\, q}\hat{L}(-\chi_1^0) < \hat{F} > \hat{L}(\chi^0). \tag{6.2.9}$$

Here

$$< \hat{F} >= \begin{pmatrix} F & (\mu_0^2 - 1)\cos 2\Psi^0 & 0 & 0 \\ (\mu_0^2 - 1)\cos 2\Psi^0 & F^+ & 0 & 0 \\ 0 & 0 & F^- & 0 \\ 0 & 0 & 0 & B\,\mu_0 \end{pmatrix}, \tag{6.2.10}$$

where

$$F^{\pm} = \frac{1}{2}[(1+\mu_0)^2\cos 4\Psi^0 \pm (1-\mu_0)^2], \tag{6.2.11}$$

and Ψ^0 is the value of Ψ for $\Phi = \Phi^0$. In particular,

$$\sin\Psi^0 = \frac{D_*}{\Delta\sqrt{r}}. \tag{6.2.12}$$

We can express all functions of Ψ^0 through that sine.

Thus, the matrix $\hat{R}$ can be presented in the form

$$\hat{R}(x, \boldsymbol{\omega} \to x_1, \boldsymbol{\omega}_1) = \hat{L}(-\chi_1^0)\hat{S}(x_1, x, \mu)\hat{L}(\chi^0), \tag{6.2.13}$$

where

$$\hat{S}(x_1,x,\mu)=\frac{3}{8}\int_{\gamma_*(x_1,x,\mu)}^{\infty} f_{\rm e}(\gamma)\mathrm{d}\gamma \hat{R}(x_1,x,\mu,\gamma). \tag{6.2.14}$$

The lower limit of the integral $\gamma_*(x_1,x,\mu)$ is the positive root of the left-hand side of inequality (6.2.6), which we shall discuss in Section 6.4.

Thus, after averaging over the azimuth in the expression for the scattering matrix, the transformation matrices $\hat{L}$ depend only on the angles between the direction of photons and vector $\mathbf{l}$. The dependence of these matrices on the electron momentum disappears. The central part of the scattering matrix is now matrix (6.2.14), depending on the photons frequencies and the scattering angle. This matrix, as well as (6.2.10), has only six non-zero elements two of which are equal because of the symmetry.

We are left to obtain the expression for matrix in the integrand (6.2.14):

$$\hat{R}(x_1,x,\mu,\gamma)=\frac{zx}{\pi q}\int_{\zeta^-}^{\zeta^+}\frac{\mathrm{d}\zeta}{D_*}<\hat{F}>. \tag{6.2.15}$$

To do this, we must calculate the integral over ζ.

It is easy to see that all the quantities in the integral (6.2.15) can be expressed through ξ and ξ_1 as μ_0 and B (in accordance with (2.3.11) and (2.3.13)). In their turn, ξ and ξ_1 are expressed through the variable ζ. All these integrals are the elementary functions. Thus, in the next section we shall give the five redistribution functions (RF) describing the redistribution on directions and frequencies of the Stokes parameters due to the scattering.

6.3 Redistribution functions

Although the calculation of the scattering matrix is performed for the scattering $\mathbf{k},\mathbf{p}\to\mathbf{k}_1,\mathbf{p}_1$, we present the result in such a form to write down the kinetic equation. For the process $\mathbf{k}_1\to\mathbf{k}$

$$\hat{R}(x_1,\boldsymbol{\omega}_1\to x,\boldsymbol{\omega}) = \hat{L}(-\chi^0)\hat{S}(x,x_1,\mu)\hat{L}(\chi_1^0), \tag{6.3.1}$$

$$\hat{S}(x,x_1,\mu) = \frac{3}{8}\int_{\gamma_*(x,x_1,\mu)}^{\infty} f_{\rm e}(\gamma)\mathrm{d}\gamma\hat{R}(x,x_1,\mu,\gamma). \tag{6.3.2}$$

After integration over the directions of the electron momenta, the scattering matrix for the isotropic electron gas with fixed energy takes the form:

$$\hat{R}(x,x_1,\mu,\gamma)=\begin{pmatrix} R & R_{\rm I} & 0 & 0\\ R_{\rm I} & R_{\rm Q} & 0 & 0\\ 0 & 0 & R_{\rm U} & 0\\ 0 & 0 & 0 & R_{\rm V}\end{pmatrix}. \tag{6.3.3}$$

The elements of the scattering matrix (6.3.3) (redistribution functions) have common parts; therefore, it is convenient to introduce a set of designations, namely:

$$R_{\rm a} = \frac{x+x_1}{q^2}\left(\frac{\gamma_-}{a_-^3}+\frac{\gamma_+}{a_+^3}\right)+\frac{r}{q}\left(\frac{1}{a_-^3}-\frac{1}{a_+^3}\right)-\frac{2}{q^2}\left(\frac{1}{a_-}-\frac{1}{a_+}\right),$$

$$
\begin{aligned}
R_{\rm b} &= \frac{2}{Q} + \frac{q-2}{q}\left(\frac{1}{a_-} - \frac{1}{a_+}\right), \\
R_{\rm c} &= \frac{2(x+x_1)}{q^2 r}\left(\frac{\gamma_-}{a_-} + \frac{\gamma_+}{a_+}\right) + \frac{4}{q^2 r}(a_- - a_+).
\end{aligned} \tag{6.3.4}
$$

Here

$$
a_\pm^2 = (\gamma_\pm)^2 + r, \tag{6.3.5}
$$

$$
\gamma_+ = \gamma + x_1, \quad \gamma_- = \gamma - x. \tag{6.3.6}
$$

Then

$$
\begin{aligned}
R &= R_{\rm a} + R_{\rm b}, \\
R_{\rm I} &= R_{\rm a} + R_{\rm c}, \\
R_{\rm U} &= \frac{2}{Q} + \frac{4\,Q\,(x+x_1)^2}{q^2 r^2} + 2\,R_{\rm c} - \frac{4}{qr}(a_- - a_+) + \frac{2}{q}\left(\frac{1}{a_-} - \frac{1}{a_+}\right) \\
&- \frac{4}{q^2 r^2}[2\,(a_-^3 - a_+^3) + 3\,(x+x_1)(\gamma_-\,a_- + \gamma_+\,a_+)], \\
R_{\rm Q} &= R_{\rm U} + R_{\rm a}, \\
R_{\rm V} &= R_{\rm b} - qR_{\rm a}.
\end{aligned} \tag{6.3.7}
$$

The expression for the function R (redistribution function of intensity to intensity, i.e. neglecting the polarization) was obtained by Jones (1968). Unfortunately, the formulae given there contain several misprints (see Coppi & Blandford, 1990). Correct expression was derived by Aharonian & Atoyan (1981) and was studied in detail in the works by Kershaw et al. (1986) and Nagirner et al. (1991). Some of its properties will be considered in Section 8.

Expressions (6.3.7) are useful to obtain various asymptotic expansions and approximate formulae (see Sect.7). However, when the frequencies x and x_1 are small, the quantites $a_\pm$ are very close to each other, and the loss of accuracy can appear in the difference between them. The same phenomenon occurs when μ is close to 1. To prevent loss of significance, we must regroup the terms in (6.3.7). Using the additional designations

$$
u = a_+ - a_- = (x+x_1)(2\gamma + x_1 - x)/(a_- + a_+), \quad v = a_-\,a_+, \tag{6.3.8}
$$

we represent the quantities (6.3.4) in the forms:

$$
\begin{aligned}
R_{\rm a} &= u\frac{(u^2 - Q^2)(u^2 + 5v)}{2q^2 v^3} + u\frac{Q^2}{q^2 v^2}, \\
R_{\rm b} &= \frac{2}{Q} + \frac{u}{v}\left(1 - \frac{2}{q}\right), \\
R_{\rm c} &= \frac{u}{vq}\left(\frac{u^2 - Q^2}{qr} - 2\right)
\end{aligned} \tag{6.3.9}
$$

The formulae for R and $R_{\rm V}$ do not demand the transformation, but

$$R_{\rm U} = \frac{2}{Q} + 2\frac{u-Q}{qr}\left[\frac{u-Q}{qr}(2Q+u) - 4\right] + \frac{2u}{vq} + 2R_{\rm c}. \qquad (6.3.10)$$

The formula for $R_{\rm Q}$ remains unchanged.

Loss of accuracy appears as well when we calculate the difference $u-Q$ for γ close to γ_*. In that case, we present this difference in terms of the two auxilliary quantities:

$$\begin{aligned} u-Q &= 2qrC_{\rm u}\, D_{\rm u}/(u+Q), \\ D_{\rm u} &= (\gamma + x_1 - x + \gamma_*)(\gamma - \gamma_*), \\ C_{\rm u} &= 2/[\gamma(\gamma + x_1 - x) + r + xx_1\mu + v]. \end{aligned} \qquad (6.3.11)$$

There is no another difficulties in calculating the redistribution functions.

6.4 Boundaries and relations

As it was mentioned above, the integrals over azimuth Φ differ from zero and, therefore RFs are not zero only if the inequality (6.2.6) satisfies. For the scattering $\mathbf{k}_1 \to \mathbf{k}$ this inequality must be written with changing $x \leftrightarrow x_1$:

$$z^2Q^2 - [\gamma(x_1 - x) - q]^2 \geq 0. \qquad (6.4.1)$$

The limits for the quantities which RFs depend on follow from this inequality. These limitations have different forms, depending on what values are fixed. If we fix x, x_1 and μ, then from (6.4.1) we have $\gamma \geq \gamma_*(x, x_1, \mu)$, where

$$\gamma_*(x, x_1, \mu) = (x - x_1 + Q\, t_*)/2, \quad t_* = \sqrt{1 + 2/q}. \qquad (6.4.2)$$

These designations and inequality were used in the expression of the integral in (6.2.14).

If we calculate RFs (6.3.7) for the fixed electron energy $\gamma \geq 1$, then, for the fixed initial photon frequency x_1 and the scattering angle $\arccos\mu$, the frequency of the scattered photon is limited by the roots of (6.4.1): $x_- \leq x \leq x_+$, where

$$x_\pm = x_1 \frac{\mu + \gamma(\gamma + x_1)w \pm z\, w\, a_+}{1 + 2\,\gamma\, x_1\, w + x_1^2 w^2}, \qquad (6.4.3)$$

and $w = 1 - \mu$. If the frequency of the scattered photon is fixed, we can obtain the limitations for the frequency of the initial photon, but they are more complex, because the coefficient at x_1^2 in (6.4.1) can be positive and negative. Namely, for $0 \leq x\, w \leq \gamma - z$, the inequalities $x_1^- \leq x_1 \leq x_1^+$ must satisfy, where

$$x_1^\pm = x\, \frac{\mu + \gamma(\gamma - x)w \pm z\, w\, a_-}{1 - 2\,\gamma\, x\, w + x^2 w^2}. \qquad (6.4.4)$$

If $x\, w$ is limited by $\gamma \pm z$, then $x_1 > x_1^-$.

Finally, for fixed x, x_1, γ the values of μ do not fill out the whole interval $[-1,\ 1]$ and we shall discuss this in Section 8.1.

Let us present here the expressions for RFs in the cases when all operations of the previous section can not be performed, namely when the co-multiplier of $\cos\Phi$ in the argument of the δ-function is zero. Then the integral over Φ is equal to $2\,\pi$ and we must calculate the integral over ζ in another way. We do not present the calculation and give at once the results.

For the case of scattering by immovable electrons

$$\hat{R}(x, x_1, \mu, \gamma)/z|_{\gamma=1} = 2\delta(x_1 - x - q)\hat{F}|_{\mu_0=\mu, x_1=x/(1-xw)}. \tag{6.4.5}$$

If the energy of the initial photon is formally zero, then the frequency of the scattered photon is zero as well, and the scattering matrix takes the form:

$$\hat{R}(x, 0, \mu, \gamma) = \delta(x) \begin{pmatrix} R^{\mathrm{x}} & R_{\mathrm{I}}^{\mathrm{x}} & 0 & 0 \\ R_{\mathrm{I}}^{\mathrm{x}} & R_{\mathrm{Q}}^{\mathrm{x}} & 0 & 0 \\ 0 & 0 & R_{\mathrm{U}}^{\mathrm{x}} & 0 \\ 0 & 0 & 0 & R_{\mathrm{V}}^{\mathrm{x}} \end{pmatrix}, \tag{6.4.6}$$

where

$$\begin{aligned} R^{\mathrm{x}} &= 4\ln(\gamma + z) + 2w\gamma z[w(\gamma^2 + z^2) - 2], \\ R_{\mathrm{I}}^{\mathrm{x}} &= -2(1-\mu^2)\gamma z, \\ R_{\mathrm{U}}^{\mathrm{x}} &= [-6\mu^2 + 4\mu + 6]\ln(\gamma + z) + \gamma z[4\mu^2 + 4\mu - 8 - 2w^2(2\gamma^2 - 5)/3], \\ R_{\mathrm{Q}}^{\mathrm{x}} &= [8\mu^2 - 4]\ln(\gamma + z) - \gamma z[4\mu^2 + 4\mu - 8 - 2w^2(2\gamma^2 - 5)/3], \\ R_{\mathrm{V}}^{\mathrm{x}} &= 4\ln(\gamma + z) - 4w\gamma z. \end{aligned} \tag{6.4.7}$$

The forward scattering is a special case as well, because the photon frequency does not change:

$$\hat{R}(x, x_1, 1, \gamma) = 4\delta(x - x_1)\ln(\gamma + z). \tag{6.4.8}$$

The scattering matrix in that case degenerates to the unit matrix. Finally, for $\mu \to -1$ all functions are continuous, function $R_{\mathrm{I}}(x, x_1, -1, \gamma) = 0$ and other RFs have finite limits.

We have derived the expressions for RFs for the scattering $x, \boldsymbol{\omega} \to x_1, \boldsymbol{\omega}_1$, but the result has been represented for the scattering $x_1, \boldsymbol{\omega}_1 \to x, \boldsymbol{\omega}$. RFs of the direct and inverse processes are related by the simple relation which can be obtained from the explicit expressions (6.3.7):

$$\hat{R}(x_1, x, \mu, \gamma) = \hat{R}(x, x_1, \mu, \gamma + x - x_1). \tag{6.4.9}$$

Analogous relations hold for all limitations and limiting cases which are considered in this section.

7 Scattering matrix for Maxwellian electrons

7.1 General case

With regards to the mean values, we consider here the case when the relativistic electron gas is thermal, i.e. it has Maxwellian distribution on energies (2.1.8). We present the results of Poutanen (1994), where in the same manner as for the scalar RF studied earlier (see Kershaw et al., 1986; Nagirner & Poutanen, 1993a) the expansions in a series for the scattering matrix in the limiting cases of weakly-relativistic and ultrarelativistic electron gas, and for small photon energies were obtained.

We present the matrix $\hat{S}$ in the same form as matrix $\hat{R}$ in (6.3.3), i.e. denote the elements of (6.3.2) by S with the same subscripts as R in (6.3.3). Other designations of S and R with the same indices correspondent to each other, i.e. related by the Eq. (6.3.2). Then the relations (6.3.7) are valid for S with the correspondent indices. Do not confuse these RFs S with S_j-functions from Sect. 3, 4 and 5.

We note that the direct consequence of relationship (6.4.9) is the relation between the scattering matrix, averaging over the Maxwellian electron distribution:

$$\hat{S}(x_1, x, \mu) = \exp\left(-y(x_1 - x)\right) \hat{S}(x, x_1, \mu). \tag{7.1.1}$$

This relation can be useful for checking the accuracy and for reducing the scope of the programmes for calculating RFs, because we can confine ourselves to the case $x \geq x_1$.

Observing that the quantities q, Q, x, x_1 and r do not depend on the electron energy and introducing the designations

$$\begin{aligned}
\delta_n^- &= \int_{\gamma_*}^{\infty} (a_-^n - a_+^n) \exp\left(-y\,(\gamma - \gamma_*)\right) \mathrm{d}\gamma, \\
\delta_n^+ &= \int_{\gamma_*}^{\infty} (\gamma_- a_-^n + \gamma_+ a_+^n) \exp\left(-y\,(\gamma - \gamma_*)\right) \mathrm{d}\gamma,
\end{aligned} \tag{7.1.2}$$

we write down the quantities (6.3.7) averaged over the Maxwellian distribution in the form

$$\begin{aligned}
S_{\mathrm{a}} &\propto \frac{x + x_1}{q^2}\,\delta_{-3}^+ + \frac{r}{q}\,\delta_{-3}^- - \frac{2}{q^2}\,\delta_{-1}^-, \\
S_{\mathrm{b}} &\propto \frac{2}{Q\,y} + \frac{q-2}{q}\,\delta_{-1}^-, \\
S_{\mathrm{c}} &\propto \frac{2(x + x_1)}{q^2 r}\delta_{-1}^+ + \frac{4}{q^2 r}\,\delta_1^-, \\
S_{\mathrm{U}} - 2\,S_{\mathrm{c}} &\propto \frac{2}{y}\left(\frac{1}{Q} + \frac{2\,Q\,(x + x_1)^2}{q^2 r^2}\right) - \frac{4}{qr}\delta_1^- + \frac{2}{q}\delta_{-1}^- \\
&- \frac{12\,(x + x_1)}{q^2 r^2}\delta_1^+ - \frac{8}{q^2 r^2}\,\delta_3^-,
\end{aligned} \tag{7.1.3}$$

and the coefficient of proportionality is $3ye^{-y\gamma_*}/32\pi K_2(y)$.

Integrating $\delta^{\pm}$ by parts, we express (7.1.2) in terms of the other four functions:

$$\begin{aligned}
\delta_{-1}^{-} &= \sigma_0^{-}, \quad \delta_{-1}^{+} = \sqrt{r}\sigma_1^{+}, \quad \delta_{-3}^{-} = \frac{qt_*}{\Pi} + \frac{y}{\sqrt{r}}\sigma_1^{-}, \\
\delta_{-3}^{+} &= \frac{(x+x_1)t_*}{\Pi} - y\,\sigma_0^{+}, \quad \delta_1^{-} = -\frac{Q}{y} + \frac{\sqrt{r}}{y}\sigma_1^{-}, \\
\delta_1^{+} &= \frac{Q\,(1+q)(x+x_1)}{yq} + \frac{2(x+x_1)t_*}{y^2} + \frac{\sqrt{r}}{y}\left(\frac{2}{y}\sigma_1^{+} - \sqrt{r}\sigma_0^{+}\right), \\
\delta_3^{-} &= -\frac{\Phi_*}{y} - \frac{3(Q^2+q\,r)t_*}{y^2} - \frac{6Q}{y^3} + \frac{3\sqrt{r}}{y^2}\left(\frac{2}{y}\sigma_1^{-} - \sqrt{r}\sigma_0^{-}\right).
\end{aligned} \tag{7.1.4}$$

Here

$$\Pi = \frac{Q^2+qr(q+2)}{2q}, \quad \Phi_* = \frac{Q}{2q}[(2q+3)Q^2+3rq(q+2)], \tag{7.1.5}$$

$$\sigma_n^{\pm} = e^{y\,\rho_-}L_n^{-} \pm e^{y\,\rho_+}L_n^{+}, \rho_{\pm} = \gamma_{\pm}(\gamma_*) = [Qt_* \pm (x+x_1)]/2, \tag{7.1.6}$$

$$L_n^{\pm} = \int_{c_\pm}^{\infty} \exp(-y_{\rm r}p)\frac{p^n \mathrm{d}p}{\sqrt{1+p^2}}, \quad y_{\rm r} = \sqrt{r}\,y, \quad c_{\pm} = \rho_{\pm}/\sqrt{r}. \tag{7.1.7}$$

The methods of calculation of integrals (7.1.7) are considered in works by Nagirner & Poutanen (1993a) and Poutanen (1994).

7.2 Ultrarelativistic limit

Let us assume that the electron temperature is large, so $y \ll 1$. We assume as well that $y_{\rm r} = \sqrt{r}y \ll 1$. By making the change of variables $p = (u - 1/u)/2$ in the integrals (7.1.7), expanding the exponents in a Taylor series and integrating by parts, one obtains the following expansions

$$L_0^{\pm} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{y_{\rm r}}{2u_{\pm}}\right)^k E_{k+1}(\frac{y_{\rm r}u_{\pm}}{2}), \quad L_1^{\pm} = -\frac{2}{y_{\rm r}}L_0^{\pm} + \frac{1}{y_{\rm r}}e^{-y\rho_{\pm}}, \tag{7.2.1}$$

where $u_{\pm} = (t_* \pm 1)(Q+x+x_1)/2\sqrt{r}$, and E_k is the integral exponential function of order k.

Thus, functions (7.1.7), and through (7.1.3), the scattering matrix (6.2.14) as well, are represented by the quickly converging series. The expansions for the RF S were used by Kershaw et al. (1986) and Nagirner & Poutanen (1993a).

7.3 Small frequencies

The following case, when the expressions of RFs for Maxwellian electrons are simplified, is that of small frequencies ($h\nu \ll mc^2$) of initial and scattered photons. Let us assume that $x, x_1 \to 0$, and the ratio $x_1/x = X$ is constant. Then for $x \to 0$ the product $x\,\hat{R}(x, X\,x, \mu, \gamma)$ has a finite limit. Expanding it in a Taylor series in x, we have:

$$x\,\hat{R}(x, X\,x, \mu, \gamma) = \sum_{n=0}^{\infty} \hat{R}_n(X, \mu, \gamma)\, x^n. \tag{7.3.1}$$

It is easy to see that the coefficients of expansion in the second argument are the same:

$$x\,\hat{R}(X\,x, x, \mu, \gamma) = \sum_{n=0}^{\infty} \hat{R}_n(X, \mu, \gamma)\,(-x)^n. \tag{7.3.2}$$

The formulae for the coefficients of power series (7.3.1) can be obtained by expanding the powers of $a_\pm$ in a series in Legendre polynomials. The initial expansions are the following:

$$\frac{1}{a_-} = \frac{1}{a_0} \sum_{n=0}^{\infty} \left(\frac{x}{a_0}\right)^n P_n\left(\frac{\gamma}{a_0}\right) \tag{7.3.3}$$

and the expansion for $1/a_+$, which can be obtaned by replacing x by $-x_1$. Here we denote $a_0^2 = \gamma^2 + r$. The coefficients of the expansion of matrix (6.3.7) are easy to find, substituting series, which follow from (7.3.3) and (7.3.1) into formulae (6.3.4) and (6.3.7)

$$\begin{aligned}
R_{n\mathrm{a}}(X, \mu, \gamma) &= \frac{1}{a_0^n}\left\{\frac{\chi_n}{X^2\,w^2 a_0^4} P_{n+3}\right.\\
&- \left.\frac{1}{X\,w\,a_0^2}[1-(-X)^{n+1}]\,\frac{(n+2)(n+3)}{2\,n+5} P_{n+3,n+1}\right\},\\
R_{n\mathrm{b}}(X, \mu, \gamma) &= \frac{2}{Q_0}\delta_{n0} + \frac{1}{a_0^n}\left\{[1-(-X)^{n-1}]\,P_{n-1} - \frac{2}{X\,w\,a_0^2}[1-(-X)^{n+1}]\,P_{n+1}\right\},\\
R_{n\mathrm{c}}(X, \mu, \gamma) &= \frac{2}{r\,X^2 w^2\,a_0^{n+2}}\,\frac{\chi_n}{2\,n+5}\,P_{n+3,n+1},\\
R_{n\mathrm{U}}(X, \mu, \gamma) &= 2\,R_{n\mathrm{c}}(X, \mu, \gamma) + \left[\frac{2}{Q_0} + \frac{4Q_0}{r^2 w^2 X^2}\,(1+X)^2\right]\delta_{n0}\\
&+ \frac{2}{X\,w}\,\frac{1}{a_0^n}\left(\frac{P_{n+1}}{a_0^2} + \frac{2P_{n+1,n-1}}{r(2\,n+1)}\right)\left[1-(-X)^{n+1} - \frac{6\chi_n}{rXw(n+2)(n+3)}\right].
\end{aligned} \tag{7.3.4}$$

Here for short we denote

$$\begin{aligned}
\chi_n &= (n+1)[1-(-X)^{n+3}] + (n+3)\,X\,[1-(-X)^{n+1}],\\
Q_0^2 &= (1-X)^2 + 2\,w\,X, \quad P_n = P_n(\gamma/a_0), \quad P_{n,m} = P_n - P_m.
\end{aligned} \tag{7.3.5}$$

The expansions of RFs are obtained by substitution of (7.3.4) for relation (6.3.7).

From the explicit expressions (7.3.4) or from (7.3.1) and (7.3.2) the property of the matrix coefficients follows

$$\hat{R}_n(1/X, \mu, \gamma) = (-1)^n X^{1-n} \hat{R}_n(X, \mu, \gamma). \tag{7.3.6}$$

Expansions (7.3.1) are not more useful than the expressions from Section 6.3 but they help us to obtain the expansions for the functions averaged over the Maxwellian distribution. Actually, substituting (7.3.1) and (7.3.4) into (6.3.2) and integrating over γ, we obtain

$$\hat{S}(x, x_1, \mu) = \frac{3}{32\,\pi}\frac{y}{K_2(y)}e^{-y\gamma_*}\sum_{n=0}^{\infty} x^{n-1}\hat{S}_n(X, \mu). \tag{7.3.7}$$

The coefficients of the expansion are deduced from (7.3.4) and (7.3.1) and expressed by means of the two sets of functions $I_n^{\pm}$

$$\begin{aligned}
S_{n\mathrm{a}}(X, \mu) &= \frac{\chi_n}{X^2 w^2} I^+_{n+3} - \frac{1}{X\,w}\frac{(n+2)(n+3)}{2\,n+5}[1-(-X)^{n+1}]I_{n+3,n+1}, \\
S_{n\mathrm{b}}(X, \mu) &= \frac{2}{yQ_0}\delta_{n0} + [1-(-X)^{n-1}]I^+_{n-1} - \frac{2}{X\,w}[1-(-X)^{n+1}]\,I^+_{n+1}, \\
S_{n\mathrm{c}}(X, \mu) &= \frac{2}{r\,X^2 w^2}\frac{\chi_n}{2\,n+5}I_{n+3,n+1}, \\
S_{n\mathrm{U}}(X, \mu) &= 2\,S_{nc} + \left[\frac{2}{Q_0} + \frac{4\,Q_0}{r^2 X^2 w^2}(1+X)^2\right]\delta_{n0} \\
&+ \frac{2}{X\,w}\left(I^+_{n+1} + \frac{2I_{n+1,n-1}}{r(2\,n+1)}\right)\left[1-(-X)^{n+1} - \frac{6\chi_n}{rXw(n+2)(n+3)}\right].
\end{aligned} \tag{7.3.8}$$

The functions mensioned above are the integrals

$$I_n^{\pm} = \int_{\gamma_*}^{\infty} e^{-y(\gamma-\gamma_*)}P_n\left(\frac{\gamma}{a_0}\right)\frac{\mathrm{d}\gamma}{a_0^{n\pm1}}, \quad I_{n,m} = I_n^- - I_m^+. \tag{7.3.9}$$

Computations of these functions can be performed by the recurrent relations

$$\begin{aligned}
n\,I_n^+ &= \frac{1}{a_*^n}P_{n-1}\left(\frac{\gamma_*}{a_*}\right) - yI^+_{n-1}, \qquad (7.3.10)\\
n\,(2\,n-3)\,I_n^- &= \frac{2\,n-1}{a_*^{n-2}}P_{n-1}\left(\frac{\gamma_*}{a_*}\right) + 2\,(n-1)I^+_{n-2} - y(2\,n-1)I^-_{n-1},
\end{aligned}$$

where $a_*^2 = \gamma_*^2 + r$. It is necessary to calculate the first two functions of each set numerically by the same procedure as for integrals (7.1.7).

The recurrent relations give accurate results for not all values of variables. For large y, errors increase very quickly, but in that case the integrals (7.3.9) are easy to calculate using the Gauss-Laguerre quadrature. The formulae of the next section can be useful as well.

7.4 Non-relativistic limit

When the temperature of the electron gas is low, the value of $y = mc^2/k_B\,T_\mathrm{e}$ is large. In that case it is possible to obtain the expansion in negative powers of y. Let us present here the terms of order to $1/y^2$ inclusively.

The initial expansions are

$$\frac{1}{a_\pm} = \frac{1}{\alpha^*_\pm}\sum_{n=0}^{\infty}\left(\frac{\gamma_* - \gamma}{\alpha^*_\pm}\right)^n P^\pm_n, \tag{7.4.1}$$

where

$$\alpha^*_\pm = a_\pm(\gamma_*) = [(x+x_1)\,t_* \pm Q]/2, \quad P^\pm_n = P_n(\rho_\pm/\alpha^*_\pm). \tag{7.4.2}$$

Substituting (7.4.1) into (6.3.4), (6.3.7) and (6.2.14), integrating by parts and collecting the terms with equal powers of y, we obtain for functions (7.1.2)

$$y\,\delta^\pm_n \approx \sum_{j=0}^{2}\Delta^\pm_{n,j}/y^j. \tag{7.4.3}$$

We express the quantities here by means of the functions introduced earlier:

$$\begin{aligned}
\Delta^-_{-1,0} &= \frac{Q}{\Pi}, \quad \Delta^-_{-1,1} = \frac{t_*}{2q\Pi^3}[-Q^4 + rq(q-1)Q^2 + r^2q^2(q+2)],\\
\Delta^-_{-1,2} &= \frac{2\Phi_*}{\Pi^3} - \frac{3r\Phi_{**}}{\Pi^5},\\
\Delta^+_{-1,0} &= \frac{Q(x+x_1)}{q\,\Pi}, \quad \Delta^+_{-1,1} = \frac{r(x+x_1)t_*}{2q\,\Pi^3}[(2q+1)Q^2 + rq(q+2)],\\
\Delta^+_{-1,2} &= -\frac{3r}{\Pi^5}\frac{Q(x+x_1)}{4q^3}[(2q^2+4q+1)Q^4\\
&+ 2rq(1+q-q^2)(q+2)Q^2 + r^2q^2(1-2q)(q+2)^2],\\
\Delta^-_{-3,0} &= \frac{\Phi_*}{\Pi^3}, \quad \Delta^-_{-3,1} = \frac{3}{\Pi^5}\frac{t_*}{4q^2}[-2(q+1)Q^6\\
&+ rq(2q^2-2q-7)Q^4 + 2r^2q^2(2q^2+3q-2)Q^2 + r^3q^3(q+2)^2],\\
\Delta^-_{-3,2} &= \frac{12\Phi_{**}}{\Pi^5} - \frac{15r}{\Pi^7}\frac{Q}{8q^3}[(8q^3+28q^2+28q+7)Q^6\\
&+ 7rq_*(2q+3)(2q+1)Q^4 + 7r^2q_*^2(4q+3)Q^2 + 7r^3q_*^3],\\
\Delta^+_{-3,0} &= \frac{Q(x+x_1)}{2q^2\,\Pi^3}[(q+1)Q^2 + r\,q_*(1-q)],\\
\Delta^+_{-3,1} &= \frac{3r(x+x_1)t_*}{4q^2\Pi^5}[(4q^2+6q+1)Q^4\\
&+ 2rq_*(3q+1)Q^2 + r^2q_*^2] - \frac{2}{r}\Delta^+_{-1,1},\\
\Delta^+_{-3,2} &= -\frac{15rQ(x+x_1)}{8q^4\Pi^7}[(q+1)(4q^2+8q+1)Q^6\\
&+ rq_*(-4q^3+4q^2+15q+3)Q^4 + r^2q_*^2(-8q^2+3q+2)Q^2\\
&+ r^3q_*^3(1-3q)] - \frac{2}{r}\Delta^+_{-1,2},\\
\Delta^-_{1,0} &= -Q, \quad \Delta^-_{1,1} = -\frac{rqt_*}{\Pi}, \quad \Delta^-_{1,2} = \frac{r\Phi_*}{\Pi^3},
\end{aligned} \tag{7.4.4}$$

$$\begin{aligned}
\Delta^{+}_{1,0} &= Q\,(x+x_1)\frac{q+1}{q}, \quad \Delta^{+}_{1,1} = (x+x_1)t_*\left(2-\frac{r}{\Pi}\right), \\
\Delta^{+}_{1,2} &= \frac{(x+x_1)Q}{2q^3\Pi^3}[Q^4 + rq(3q+5)Q^2 + 3r^2q^2(q+2)], \\
\Delta^{-}_{3,0} &= -\Phi_*, \quad \Delta^{-}_{3,1} = -3t_*(Q^2+rq), \quad \Delta^{-}_{3,2} = -3Q\left(2+\frac{r}{\Pi}\right).
\end{aligned}$$

Here $q_* = q(q+2)$, Φ_* is defined in Eq. (7.1.5), and

$$\Phi_{**} = \frac{Q}{2q^2}[(4q^2+10q+5)Q^4 + 10rq_*(q+1)Q^2 + 5r^2q_*^2]. \tag{7.4.5}$$

Some approximate representations of RF of the intensity for Maxwellian electron distribution we give in Section 9.

7.5 Monochromatic photon beam

In this section we give an example of calculations of the scattering matrix for Maxwellian electron distribution. We consider the scattering of the monochromatic photon beam of energy $0.5keV$ on the electrons with a temperature $25keV$. For the Stokes paramaters of the initial radiation we take the following form $\tilde{n}_0 \propto (i_0, q_0, u_0, v_0)^{\mathrm{T}}$. Then the Stokes vector of the scattered radiation $\tilde{n}_1(x) \propto \hat{S}(x, x_0, \mu)\tilde{n}_0$.

First of all, we are interested in polarization in that region of angles and frequencies where the intensity is large. In Fig. 4 we show the dependence of the intensity of scattered radiation as a function of scattering angle and photon energy for the case of unpolarized initial beam.

To show the differences between polarization in the Compton and Thomson scattering, we consider four different cases of the polarization of the initial photon beam.

Unpolarized initial beam. For the Stokes vector of the initial radiation we take $i_0 = 1, \quad q_0 = u_0 = v_0 = 0$. The polarization degree of the scattered radiation is $P_{\mathrm{a}} = S_{\mathrm{I}}/S$. It differs about 20% from the polarization in the case of the Thomson scattering ($P_{\mathrm{a}}^{\mathrm{T}} = (\mu^2-1)/(1+\mu^2)$) (Fig. 5).

Linear polarization parallel to the plane of scattering. In this case $i_0 = q_0 = 1, u_0 = v_0 = 0$. The degree of polarization is $P_{\mathrm{b}} = (S_{\mathrm{Q}} + S_{\mathrm{I}})/(S + S_{\mathrm{I}})$ (Fig. 6). In this case the polarization degree differ from the polarization due to the Thomson scattering ($P_{\mathrm{b}}^{\mathrm{T}} = 1$) many times in a wide interval of energies and angles.

Linear polarization at $\alpha = 45^o$ with the plane of scattering. In this case $i_0 = u_0 = 1, q_0 = v_0 = 0$ and $\tilde{n}_1(x) \propto (S, S_{\mathrm{I}}, S_{\mathrm{U}}, 0)^{\mathrm{T}}$. The degree of polarization is $P_{\mathrm{c}} = \sqrt{S_{\mathrm{I}}^2 + S_{\mathrm{U}}^2}/S$ (Fig. 7). The differences from the Thomson scattering ($P_{\mathrm{c}}^{\mathrm{T}} = 1$) are about 20%.

Circular polarization. In this case $i_0 = v_0 = 1, q_0 = u_0 = 0$. The polarization degree is $P_{\mathrm{d}} = S_{\mathrm{V}}/S$ and there are no essential differences from $P_{\mathrm{d}}^{\mathrm{T}} = 2\mu/(1+\mu^2)$ for that electron energy. Therefore, we do not represent here the results of calculation.

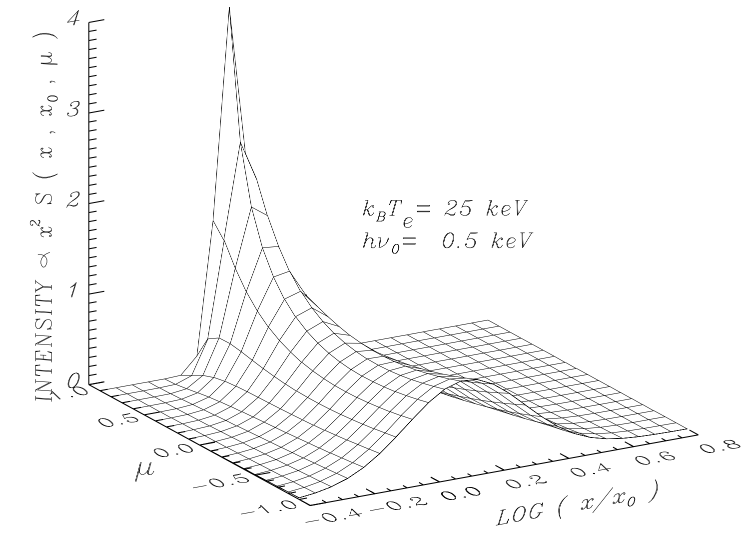

Figure 4: Intensity of the scattered radiation (arbitrary units) as a function of the cosine of the scattering angle and photon frequency.

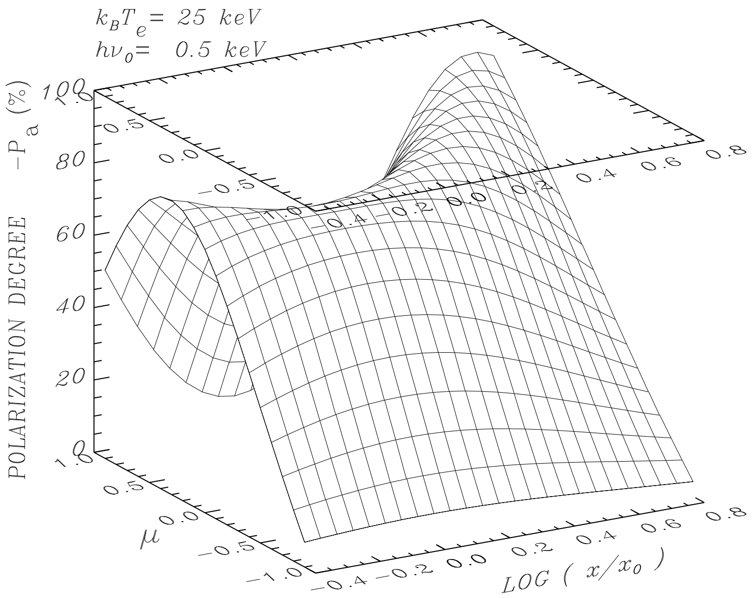

Figure 5: Polarization of scattered radiation. The case of unpolarized initial beam.

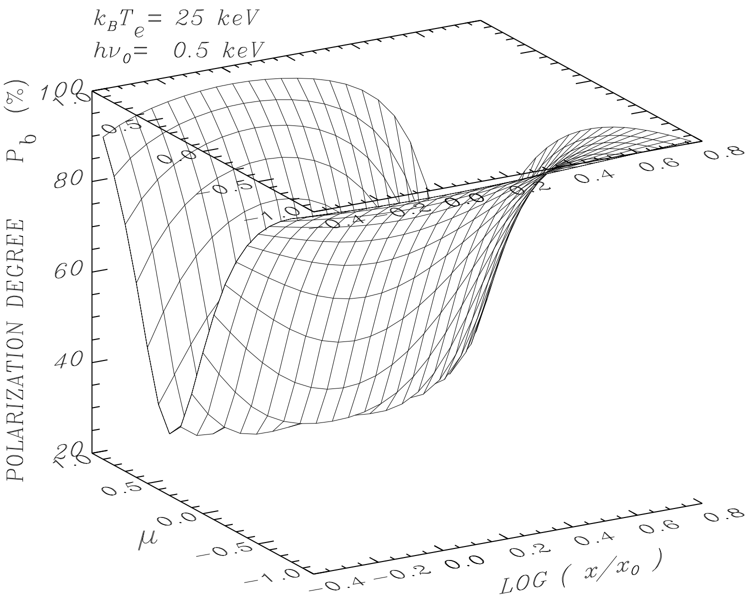

Figure 6: Same as Fig. 5, but for the case of the linear polarization parallel to the plane of scattering. For the Thomson scattering (Rayleigh matrix) $P = 100\%$.

8 Redistribution function (RF) of intensity

8.1 Averaging over scattering angle

In this section we give additional results on the RF of intensity. In according with (6.2.13) this function

$$R(x_1, \boldsymbol{\omega}_1 \to x, \boldsymbol{\omega}) = S(x, x_1, \mu) = \frac{3}{8} \int_{\gamma_*(x,x_1,\mu)}^{\infty} f_{\mathrm{e}}(\gamma) \mathrm{d}\gamma R(x, x_1, \mu, \gamma), \tag{8.1.1}$$

because the transformation of the Stokes parameters from one basis to another does not concern the intensity. The formula for $R(x, x_1, \mu, \gamma)$ can be obtained from (6.3.7) and (6.3.4) and has the form

$$\begin{aligned} R(x, x_1, \mu, \gamma) \;\; &= \;\; \frac{2}{Q} + \frac{q^2 - 2\,q - 2}{q^2}\left(\frac{1}{a_-} - \frac{1}{a_+}\right) \\ &+ \;\; \frac{1}{q^2}\left(\frac{(x + x_1)\gamma_- + rq}{a_-^3} + \frac{(x + x_1)\gamma_+ - rq}{a_+^3}\right). \end{aligned} \tag{8.1.2}$$

In many cases when there we have little information about the astrophysical object (polarization is not observed; radiation comes from the whole object, not from its parts), to interpretate such observations in the first approximation we can assume that the intensity

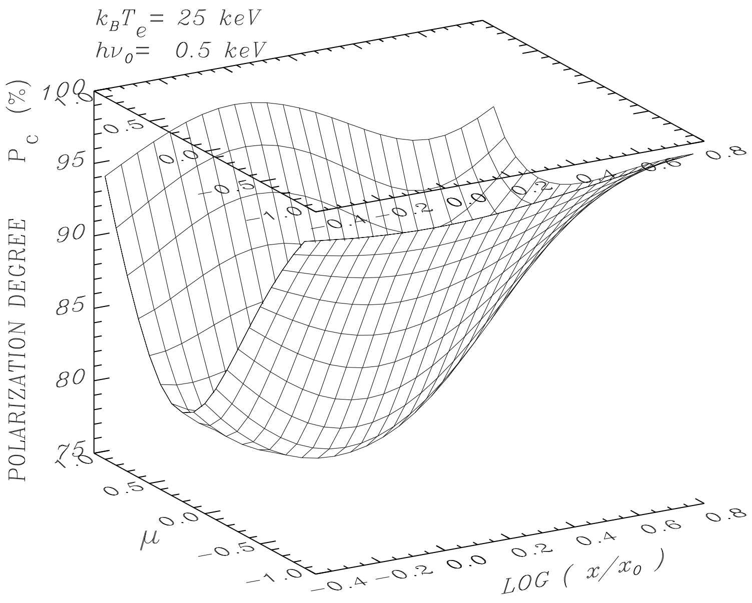

Figure 7: Same as Fig. 5, but for the case of the linear polarization at the angle 45° with the plane of scattering.

does not depend on the direction. Then to obtain the intensity of the scattered radiation we need only the redistribution function of intensity averaged over the directions. Let us find this function.

We define the RF averaged over the angles by the integral

$$S(x, x_1) = \frac{1}{2} \int_{-1}^{1} S(x, x_1, \mu)\, \mathrm{d}\mu. \tag{8.1.3}$$

To avoid the calculation of the double integral (over γ and μ), we substitute the expression (8.1.1) for (8.1.3) and change the order of integrations. Then the lower limit of the integral over γ depends only on the frequencies x and x_1. The investigation of the dependence of $\gamma_*(x, x_1, \mu)$ on μ in the interval $[-1,\ 1]$ leads us to the conclusion that the limit in the integral over γ is

$$\gamma_*(x, x_1) = \begin{cases} \gamma_*(x, x_1, -1), & \text{for} \quad |x - x_1| \geq 2\, x\, x_1, \\ \gamma_{\min}, & \text{for} \quad |x - x_1| \leq 2\, x\, x_1. \end{cases} \tag{8.1.4}$$

It is easy to obtain

$$\gamma_*(x, x_1, -1) \;=\; [x - x_1 + (x + x_1)\sqrt{1 + 1/x x_1}\,]/2, \tag{8.1.5}$$

$$\gamma_{\min} \;=\; 1 + (x - x_1 + |x - x_1|)/2. \tag{8.1.6}$$

The quantity $\gamma_*(x, x_1, \mu)$ has its minimum value $\gamma_{\min}$ at $\mu = \mu_{\min} = 1 - |x - x_1|/xx_1$. Correspondingly, the limits of integration over μ depend on the frequency and the electron energy. Investigation of these limits leads us to the following formula for (8.1.3) (see details in the paper by Nagirner et al., 1991):

$$S(x, x_1) = \frac{3}{16} \int_{\gamma_*(x,x_1)}^{\infty} \overline{R}(x, x_1, \gamma)\, f_{\rm e}(\gamma)\, {\rm d}\gamma, \tag{8.1.7}$$

where

$$\overline{R}(x, x_1, \gamma) = \int_{\mu_{\rm m}}^{\mu_+} R(x, x_1, \mu, \gamma)\, {\rm d}\mu = T(x, x_1, \mu, \gamma)\Big|_{\mu_{\rm m}}^{\mu_+}. \tag{8.1.8}$$

Here we introduce a set of designations. The limit on μ are defined by

$$\mu_{\rm m} = \begin{cases} -1, & \text{if } |x - x_1| \geq 2\, x\, x_1, \\ -1, & \text{if } |x - x_1| \leq 2\, x\, x_1 \text{ and } \gamma \geq \gamma_*(x, x_1, -1), \\ \mu_-, & \text{if } |x - x_1| \leq 2\, x\, x_1 \text{ and } \gamma_{\min} \leq \gamma \leq \gamma_*(x, x_1, -1). \end{cases} \tag{8.1.9}$$

The quantities $\mu_\pm$ are the roots of the equation $\gamma_*(x, x_1, \mu) = \gamma$:

$$\mu_- = 1 - \frac{D_{\rm m}}{x\, x_1}, \quad \mu_+ = 1 - \frac{(x - x_1)^2}{D_{\rm m}}, \tag{8.1.10}$$

where

$$D_{\rm m} = z^2 + \gamma\,(x_1 - x) + z\sqrt{z^2 + 2\,\gamma\,(x_1 - x) + (x_1 - x)^2}. \tag{8.1.11}$$

These limits were discussed in Section 6.4.

The primitive for $R(x, x_1, \mu, \gamma)$ as a function of μ can be expressed in terms of functions of one argument (Brinkmann, 1984):

$$\begin{aligned} T(x, x_1, \mu, \gamma) \;&=\; -\frac{2}{xx_1}Q + \sqrt{\frac{w}{2}}\left\{\frac{4}{xx_1}H_0 + w\left(1 + \frac{1}{xx_1}\right)H_1\right. \\ &+\; \left.\frac{H}{A(h_-)A(h_+)}\left[w + \frac{1}{2x^2x_1^2}\left(2\frac{H^2}{w} - (x - x_1)^2\right)\right]\right\}. \end{aligned} \tag{8.1.12}$$

The quantities H are the differences

$$H = A(h_-) - A(h_+), \quad H_n = A_n(h_-) - A_n(h_+). \tag{8.1.13}$$

The functions A_n are defined by the common formula by means of derivatives of the zeroth function:

$$A_n(h) = (-2)^n \frac{|2n - 1|}{(2n - 1)!!} A_0^{(n)}(h). \tag{8.1.14}$$

They can be obtained by the recurrent relation

$$A_{n+1}(h) = \frac{1}{h}\left[\frac{2n + 1}{|2n - 1|}A_n(h) - \frac{1}{A^{2n+1}(h)}\right]. \tag{8.1.15}$$

The zeroth function of this set, as well as all others, is defined by the different formulae for the positive and negative argument

$$A_0(h) = \begin{cases} \ln(\sqrt{h} + \sqrt{1+h})/\sqrt{h}, & h \geq 0, \\ \arcsin(\sqrt{-h})/\sqrt{-h}, & h \leq 0, \end{cases} \tag{8.1.16}$$

but all of them have the unique expansion in a series, which converges for $|h| \leq 1$:

$$A_n(h) = \frac{|2n-1|}{(2n-1)!!} \sum_{k=0}^{\infty} \frac{(2n+2k-1)!!}{(2k)!!} \frac{(-h)^k}{2n+2k+1}. \tag{8.1.17}$$

Besides that, following quantities enter (8.1.12):

$$h_{\pm} = (\gamma_{\pm}^2 - 1)\, w/2 \tag{8.1.18}$$

and the function without indices, the values of which are expressed by means of the quantities $a_{\pm}$ used earlier (see Eq. 6.3.5)

$$A(h_{\pm}) = \sqrt{w/2}\; a_{\pm} = \sqrt{1 + h_{\pm}}. \tag{8.1.19}$$

Note that h_+ is always positive, but h_- can be negative and $h_- \geq -w/2 \geq -1$.

The terms in (8.1.12) are disposed in such a manner that there is no loss of significance for h_- and h_+ close to one another (that happens, for example, when x and x_1 are small). If $h_- \approx h_+$, the subtraction in (8.1.13) is better performed analytically, namely:

$$H = \Delta h/[A(h_-) + A(h_+)], \;\; \Delta h = h_- - h_+ = -(2\gamma + x_1 - x)\,(x + x_1)\, w/2. \tag{8.1.20}$$

In the same manner, when $h_{\pm} \geq 0$ and $h_- \approx h_+$ are not small,

$$\begin{aligned} H_0 &= -\frac{\Delta h}{\sqrt{h_-} + \sqrt{h_+}} \frac{A_0(h_-)}{\sqrt{h_+}} + \frac{h_*}{\sqrt{h_+}} A_0(h_*^2), \\ h_* &= \frac{\Delta h}{\sqrt{h_-}A(h_+) + \sqrt{h_+}A(h_-)}, \end{aligned} \tag{8.1.21}$$

$$H_1 = [H_0 - \Delta h\, A(h_+) + H/A(h_-)A(h_+)]/h_+ \tag{8.1.22}$$

The negative h_- can be close to the positive h_+, if both are small. Then such subtraction can be performed in series. It is possible to symmetrize the differences H_0 and H_1 relative to h_- and h_+.

8.2 Averaging with weight μ

The intensity of radiation is often presented as a linear function of the cosine of angle between some directions (for example, in the case of the plane geometry between a normal to the layers and a direction of the propagation of radiation). In that case it is necessary to know the first angular moment of the redistribution function on μ. It is convenient to find the following quantity:

$$
\begin{aligned}
S^*(x,x_1) &= \frac{1}{2}\int_{-1}^{1} w\, S(x,x_1,\mu)\,\mathrm{d}\mu = \\
&= \frac{3}{16}\int_{\gamma_*(x,x_1)}^{\infty} f_{\mathrm{e}}(\gamma)\,\mathrm{d}\gamma \int_{\mu_{\mathrm{m}}}^{\mu_+} w\, R(x,x_1,\mu,\gamma)\,\mathrm{d}\mu. \qquad (8.2.1)
\end{aligned}
$$

The moment sought is the difference $S(x,x_1)-S^*(x,x_1)$. The function $S(x,x_1)$ and the difference mentioned above are the coefficients before the zeroth and first polynomials in the expansion of the redistribution function in Legendre polynomials.

For the internal integral in (8.2.1), i.e. for the fixed electron energy γ, as above, we introduce the primitive function T^*:

$$
\overline{R^*}(x,x_1,\gamma) = \int_{\mu_{\mathrm{m}}}^{\mu_+} w\, R(x,x_1,\mu,\gamma)\,\mathrm{d}\mu = T^*(x,x_1,\mu,\gamma)\Big|_{\mu_{\mathrm{m}}}^{\mu_+}. \qquad (8.2.2)
$$

It is expressed in terms of the functions used earlier:

$$
\begin{aligned}
T^*(x,x_1,\mu,\gamma) &= \frac{2}{3}\frac{Q}{x^2x_1^2}\left[(x-x_1)^2-q\right] + 2\left(\frac{w}{2}\right)^{3/2}\frac{H}{A(h_-)A(h_+)}\left(\frac{w}{2}-\frac{2}{x\,x_1}\right) \\
&- \left(1+\frac{2}{x\,x_1}\right)\left(\frac{w}{2}\right)^{5/2}\left[H\frac{A^2(h_-)+A(h_-)A(h_+)+A^2(h_+)}{A^3(h_-)A^3(h_+)} + H_2\right] \qquad (8.2.3) \\
&- \frac{1}{x\,x_1}\left(\frac{w}{2}\right)^{3/2} H_1\left[4-\frac{(x-x_1)^2}{x\,x_1}\right] + \frac{1}{x^2x_1^2}\left(\frac{w}{2}\right)^{1/2} H_{01},
\end{aligned}
$$

where

$$
H_{01} = 4\,H_0 + \Delta h\,[A_1(h_-)+A_1(h_+)]. \qquad (8.2.4)
$$

For close h_- and h_+ the quantity H_2 can be found simply by the recurrent relation

$$
H_{n+1} = \frac{1}{h_+}\left[\frac{2\,n+1}{|2\,n-1|}H_n - \Delta h\,A_{n+1}(h_-) + H\sum_{k=0}^{2\,n} A^{-k-1}(h_-)\,A^{k-2\,n-1}(h_+)\right]. \qquad (8.2.5)
$$

It is more difficult to prevent the loss of accuracy in the expression for the quantity H_{01}, which for small Δh is of order $(\Delta h)^3$. We combine the terms in a special way; thus expression (8.2.4) takes the form

$$
\begin{aligned}
H_{01} &= \frac{\tilde h\,A_0(h_-)-\sqrt{h_+}\,H_0}{2\,h_-^{3/2}h_+^{3/2}}\frac{(\Delta h)^3}{\tilde h}\left(1+\frac{\sqrt{h_-\,h_+}}{\tilde h^2}\right) \\
&+ 2\frac{\tilde h}{\sqrt{h_-\,h_+}}h_*\left[A_0(h_*^2)-1\right] - \frac{h_*(\Delta h)^2}{2h_-^{3/2}h_+^{3/2}}A_0(h_*^2)\frac{\tilde h^2-\sqrt{h_-\,h_+}}{\tilde h} \\
&- \frac{h_*^2\,H}{h_-\,h_+}\left\{\frac{\left[A(h_-)A(h_+)+\sqrt{h_-\,h_+}\right]^2}{A(h_-)A(h_+)}+1\right\}, \qquad (8.2.6)
\end{aligned}
$$

where $\tilde{h} = \sqrt{h_-} + \sqrt{h_+}$.

As above, for small $h_\pm$ we use the series expansions. In particular, for small and not small $h_\pm$ the quantity H_{01} is represented by the series:

$$H_{01} = 2(\Delta h)^3 \sum_{k=0}^{\infty} \frac{(-1)^k}{2\,k+7} \frac{(2\,k+5)!!}{(2\,k+6)!!} \sum_{m=0}^{k} (m+1)\,(k+1-m)\,h_-^{k-m}\,h_+^m, \tag{8.2.7}$$

$$H_{01} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(2\,n+3)!!(\Delta h)^{n+3}}{(2\,n)!!(n+2)(n+3)} A_{n+3}(h_-). \tag{8.2.8}$$

8.3 Relations between RFs and mean values

In this section we find the relations between the mean values studied in Sect. 3, and RF. First of all, let us consider the scattering by the isotropic electron gas of the fixed energy.

The comparison of formulae (3.1.8) and (6.1.3) gives the expression for functions Ψ_j through RF:

$$\Psi_j(x,\gamma) = \frac{3}{32\,\pi\,\gamma\,z\,x^{j+1}} \int x_1^{j+1} \mathrm{d}x_1 \mathrm{d}^2\omega_1 R(x_1,x,\mu,\gamma). \tag{8.3.1}$$

The integral over directions of the scattered photon momenta can be calculated, using the RF averaged over direction. To do this we must find the limits of variation x_1 as a function of x and γ, i.e. invert the dependence of the low limit (8.1.4) in (8.1.7). We obtain

$$\Psi_j(x,\gamma) = \frac{3}{16\,\gamma\,z\,x^{j+1}} \int_{x^-(x,\gamma)}^{x_{\mathrm{m}}(x,\gamma)} x_1^{j+1} \mathrm{d}x_1 \overline{R}(x_1,x,\gamma), \tag{8.3.2}$$

where the limits of integration

$$x_{\mathrm{m}}(x,\gamma) = \begin{cases} \gamma + x - 1, & \text{if } 1 \le \gamma \le 1 + \dfrac{2x^2}{1-2\,x} \text{ and } x < 1/2, \\ x^+(x,\gamma), & \text{if } 1 + \dfrac{2x^2}{1-2\,x} \le \gamma < +\infty \text{ and } x < 1/2, \\ \gamma + x - 1, & \text{if } x \ge 1/2, \end{cases} \tag{8.3.3}$$

$$x^\pm(x,\gamma) = x\,(\gamma \pm z)\,(\gamma \pm z + 2\,x)/(1 + 4\,\gamma\,x + 4\,x^2). \tag{8.3.4}$$

It follows from (3.1.9), (6.1.3) and (8.1.1), that the quantities averaged over the Maxwellian distribution are connected by the relation

$$\overline{x_1^j}\,\overline{s}_0(x) = (4\,\pi/x) \int_0^\infty x_1^{j+1} \mathrm{d}x_1\, S(x_1,x). \tag{8.3.5}$$

In the same manner we obtain the relations for the mean values of $x_1(1-\mu)$:

$$\Psi_1^*(x,\gamma) = \frac{3}{16\,\gamma\,z\,x^2} \int_{x^-(x,\gamma)}^{x_{\mathrm{m}}(x,\gamma)} x_1^2 \mathrm{d}x_1 \overline{R^*}(x_1,x,\gamma), \tag{8.3.6}$$

$$\overline{x_1\,(1-\mu)}\,\overline{s}_0(x) = \frac{4\,\pi}{x} \int_0^\infty x_1^2 \mathrm{d}x_1\, S^*(x_1,x). \tag{8.3.7}$$

The definitions of Ψ_j and $\overline{x^j}$ in Sect. 3 and given in this section present these quantities as a multiple integral, where the integration over variables is performed in different order, thus the deduction of Eqs. (8.3.2), (8.3.5), (8.3.6) and (8.3.7) reduces to the change of the order of integration. These relations can be the tool to check the accuracy of calculation of mean values and RF.

8.4 Zeroth and first moments of RKE

Using relations found between RF and mean values, we can write the moment equations, which follow from the relativistic kinetic equation (RKE) if we integrate it over the photon momenta with some weights. The moment equations are widely used in the kinetic theory of gases, for example, for deducing of the gas dymamic equations from the Boltzmann equation, and in the theory of radiative transfer for the approximate solution of the transfer equations (see Sobolev, 1985). Here we obtain the two first moments of our kinetic equation.

Assuming $\underline{k} = \{k, \mathbf{k}\} = m\,c\{x, x\,\boldsymbol{\omega}\}$ and taking into account the induced scattering, we write the equation for the Compton scattering in the frame E in the dimensional form and do not take into account the polarization:

$$\underline{k}\cdot\underline{\nabla} n(\mathbf{k}) = \frac{\sigma_{\mathrm{T}}\, n_{\mathrm{e}}}{m\,c}\int \frac{\mathrm{d}^3 k_1}{k_1}\,[S(x,x_1,\mu)\,n(\mathbf{k}_1)(1+n(\mathbf{k}))-S(x_1,x,\mu)\,n(\mathbf{k})(1+n(\mathbf{k}_1)]. \quad (8.4.1)$$

Multiplying (8.4.1) by $2\,c\mathrm{d}^3k/k(2\,\pi\,\hbar)^3$, we integrate it over photon momenta. Then in the integrand we obtain the expression symmetrical relative to the change $\mathbf{k} \leftrightarrow \mathbf{k}_1$, which means that the equation of continuity of the photon number holds:

$$\underline{\nabla}\cdot\underline{H}_{\mathrm{ph}} = \frac{\partial \rho_{\mathrm{ph}}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{H}_{\mathrm{ph}} = 0. \quad (8.4.2)$$

Here the four-dimensional vector of the photon flux

$$\underline{H}_{\mathrm{ph}} = \{c\,\rho_{\mathrm{ph}}, \mathbf{H}_{\mathrm{ph}}\} = \frac{2\,c}{(2\,\pi\,\hbar)^3}\int \underline{k}\,n(\mathbf{k})\,\frac{\mathrm{d}^3k}{k}, \quad (8.4.3)$$

and ρ_{ph} is the density of photons, $\mathbf{H}_{\mathrm{ph}}$ is their three-dimensional flux.

The moment of the first order can be obtained in the same manner. Let us define the density and the radiative flux of a photon field, and the tensor of pressure:

$$\rho = \frac{2\,c}{(2\,\pi\,\hbar)^3}\int k n(\mathbf{k})\,\mathrm{d}^3k, \quad \mathbf{H} = \frac{2\,c^2}{(2\,\pi\,\hbar)^3}\int \mathbf{k}\,n(\mathbf{k})\,\mathrm{d}^3k, \quad (8.4.4)$$

$$\overset{\circ}{T} = \frac{2\,c}{(2\,\pi\,\hbar)^3}\int k\,\boldsymbol{\omega}\circ\boldsymbol{\omega}\,n(\mathbf{k})\,\mathrm{d}^3k. \quad (8.4.5)$$

The definition $\boldsymbol{\omega}\circ\boldsymbol{\omega}$ represents the dyad product of the vector $\boldsymbol{\omega}$ by itself. The radiation pressure averaged over directions is expressed through the trace of the tensor

$$P = \frac{1}{3}\mathrm{Sp}\,\overset{\circ}{T} = \frac{2\,c}{3\,(2\,\pi\,\hbar)^3}\int k\,n(\mathbf{k})\,\mathrm{d}^3k = \frac{1}{3}\rho. \quad (8.4.6)$$

The last relation is the ordinary connection between the pressure and the energy density of the ultrarelativistic gas. The quantities $\rho, \mathbf{H}$ and $\overset{\circ}{T}$ are the components of the four-dimensional tensor of the radiation energy-momentum-stress flux $\overset{\diamond}{T}$.

Multiplying (8.4.1) by $(2\,c^2/(2\,\pi\,\hbar)^3)\,\underline{k}\,\mathrm{d}^3k/k$ and integrating over momenta, we obtain on the left-hand side the vector of the divergence of tensor $\overset{\diamond}{T}$:

$$\underline{\nabla}\,\overset{\diamond}{T}= \left\{\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{H},\;\; \frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}+c\,\boldsymbol{\nabla}\cdot\overset{\circ}{T}\right\}. \tag{8.4.7}$$

Making the change $\mathbf{k}\leftrightarrow\mathbf{k}_1$ in one of the terms on the right hand side, we find

$$\underline{\nabla}\,\overset{\diamond}{T}= \frac{2\,c^2\,\sigma_{\mathrm{T}}\,n_{\mathrm{e}}}{m\,c\,(2\,\pi\,\hbar)^3}\int\frac{\mathrm{d}^3k}{k}\,\frac{\mathrm{d}^3k_1}{k_1}\,S(x_1,x,\mu)\,n(\mathbf{k})[1+n(\mathbf{k}_1)](\underline{k}_1-\underline{k}). \tag{8.4.8}$$

Omitting the non-linear terms, we obtain in the linear approximation the equation of energy conservation:

$$\begin{aligned}\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{H} &= \frac{2}{(2\,\pi\,\hbar)^3}\,\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,m^4\,c^6\int\frac{\mathrm{d}^3x}{x}n(\mathbf{k})\,4\,\pi\int x_1\mathrm{d}x_1\,(x_1-x)\,S(x_1,x)= \\ &= \frac{2}{(2\,\pi\,\hbar)^3}\,\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,m^4\,c^6\int\mathrm{d}^3x\,n(\mathbf{k})\,(\overline{x_1}-x)\,\overline{s}_0(x),\end{aligned} \tag{8.4.9}$$

and also the law of the momentum conservation of the photon gas:

$$\begin{aligned}\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}+c\,\boldsymbol{\nabla}\cdot\overset{\circ}{T}&= \frac{2}{(2\,\pi\,\hbar)^3}\,\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,m^4\,c^6 \\ &\times\int\frac{\mathrm{d}^3x}{x}n(\mathbf{k})\,4\,\pi\,\boldsymbol{\omega}\int x_1\mathrm{d}x_1\,[(x_1-x)\,S(x_1,x)-x_1\,S^*(x_1,x)] \\ &= \frac{2}{(2\,\pi\,\hbar)^3}\,\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,m^4\,c^6\int\mathrm{d}^3x\,n(\mathbf{k})\,\boldsymbol{\omega}\,[(\overline{x_1}-x)-\overline{x_1(1-\mu)}]\,\overline{s}_0(x).\end{aligned} \tag{8.4.10}$$

Similar moment equations were deduced by Ochelkov et al. (1979). We have reproduced them here to show the physical sense of the mean values and averaged RFs. This explanes why we use here the dimensional quantities.

9 Approximate forms of RF and kinetic equation

9.1 Scattering on weakly-relativistic and cold electrons

In this section we shall give the limiting expressions for RF in the case when Maxwellian electrons are weakly-relativistic or even immovable, and specify some results of Sect. 7.

The exact asymptotic expansion for $S(x,x_1,\mu)$ can be obtained if $y\gg 1$

$$S(x,x_1,\mu)=\frac{3}{32\,\pi\,K_2(y)}e^{-y\,\gamma_*}\sum_{n=0}^{\infty}y^{-n}S_n(x,x_1,\mu). \tag{9.1.1}$$

Here

$$\begin{aligned} S_n(x,x_1,\mu) &= \frac{2}{Q}\delta_{n0} - \frac{n!}{q^2}\left[\frac{\theta_n\, P_n^- + (n+1)qt_* P_{n+1}^-}{-(a_-)^{n+1}}\right. \\ &+ \left.\frac{-\theta_n\, P_n^+ + (n+1)qt_* P_{n+1}^+}{-(a_+)^{n+1}}\right], \end{aligned} \tag{9.1.2}$$

where $\theta_n = q^2 - 2 + (n-1)q$. Nagirner & Poutanen (1993a) have shown how to prevent the loss of significance using this formula.

For large y the function $S(x,x_1,\mu)$ is not small only for those parameters x, x_1 and μ, for which γ_* is close to 1. We present the difference $\gamma_* - 1$ in the form

$$\gamma_*(x,x_1,\mu) - 1 = (q + x - x_1)^2/d^2, \tag{9.1.3}$$

where $d^2 = q(Q\,t_* + 2 + x_1 - x)$. Equality $q = x_1 - x$ means that the scattering is being undergone by the electron at rest. In that case

$$x_1 = x/(1 - x\,w), \;\; w = 1 - \mu = 1/x - 1/x_1, \tag{9.1.4}$$

i.e. x, x_1, μ coincide with ξ, ξ_1, μ_0 respectively. The function $S(x,x_1,\mu)$ has a δ-function character. Let us show it explicitly.

To deduce the limiting form of the equation, we use the following expansion of exponent in δ-function and its derivatives

$$\sqrt{\frac{y}{\pi}} e^{-y\,\chi^2} = \sum_{n=0}^{\infty} \frac{\delta^{(2\,n)}(\chi)}{n!(4\,y)^n} \sim \delta(\chi) + \frac{1}{4\,y}\delta''(\chi), \tag{9.1.5}$$

which is easy to verify if we find the Fourier transforms on χ from both sides of the equality. We shall take into account later only terms of order $1/y$.

Let us write Eq. (9.1.1) with the same accuracy

$$S(x,x_1,\mu) \sim \frac{3}{32\,\pi\,K_2(y)} e^{-y\gamma_*}[S_0(x,x_1\mu) + \frac{1}{y}\,S_1(x,x_1,\mu)], \tag{9.1.6}$$

where

$$S_0(x,x_1,\mu) = 2\frac{(1+q)Q^4 + q^2(q+2)rQ^2 + q^2r^2(q+2)^2}{Q[Q^2 + qr(q+2)]^2}, \tag{9.1.7}$$

$$S_1(x,x_1,\mu) = \frac{4q^2t_*^3[(q^2-4)q^3r^3 + (q^2-2q+12)q^2r^2Q^2 + 5qrQ^4 - Q^6]}{[Q^2 + qr(q+2)]^4}. \tag{9.1.8}$$

Using substitutions (9.1.4) simplifies functions S_n:

$$\begin{aligned} S_0\,|_{x_1=x+q} &= \frac{F_*}{\sqrt{q\,(q+2)}}, \;\; S_1\,|_{x_1=x+q} = \frac{w\,F_{1*}}{[q\,(q+2)^3]^{1/2}}, \;\; F_* = 1 + \mu^2 + w\,q, \\ F_{1*} &= 2(2 + \mu - 5\mu^2) + \mu(2 - 3\,\mu + 3\,\mu^2)q + \mu\,w\,q^2, \end{aligned} \tag{9.1.9}$$

because now $Q^2 = q(q+2)$, $d = \sqrt{2}Q$. The asymptotic properties of the Macdonald function allow us to write RF in the form

$$S(x, x_1, \mu) = \frac{3}{16\,\pi}\left\{\delta(x_1 - x - q)\left[F_*\left(1 - \frac{15}{8\,y}\right) + \frac{wF_{1*}}{y(q+2)}\right]\right.$$
$$\left. + \frac{S_0}{4\sqrt{2}y}\,\delta''\left(\frac{x_1 - x - q}{d}\right)\right\}. \qquad (9.1.10)$$

This representation is the initial step in deducing various approximate expressions. For cold electrons, i.e. for $y = \infty$ the exact limiting form of the kinetic equation can be obtained from (8.4.1) and (9.1.10):

$$\frac{1}{c}\frac{\partial n}{\partial t} + \boldsymbol{\omega}\cdot\boldsymbol{\nabla} n = \frac{3}{16\,\pi}\sigma_{\rm T}\,n_{\rm e}\int\frac{x_1^2}{x^2}{\rm d}^2\omega_1$$
$$\times\ \left[n_{11}(1+n)\,F_*|_{x_1=x/(1-xw)} - n(1+n_{11})\,F_*|_{x_1=x/(1+xw)}\right]. \qquad (9.1.11)$$

Here $n_{11} = n(x_1, \boldsymbol{\omega}_1)$ (dependence on coordinates and time is omitted).

For the case when $n = n(t, x)$ does not depend both on the coordinates and on the direction of the vector $\boldsymbol{\omega}$, i.e. the radiation field is isotropic and homogeneous, an equation like (9.1.11) and reduced from it the differential Fokker-Planck equation for small frequencies were obtained in the paper by Ross et al. (1978) (see also Nagirner, 1984).

Let us consider in detail the case when the energies of the non-relativistic electrons and soft photons are of the same order ($\sim k_{\rm B}\,T_{\rm e} \ll m\,c^2$).

9.2 Scattering of soft photons by nearly non-relativistic electrons

In that case it is convenient to measure the photon frequency not in units of electron rest energy, but in units of thermal energy $k_{\rm B}\,T_{\rm e}$. We put $x_{\rm t} = x\,y = k\,c/k_{\rm B}\,T_{\rm e}$. Then

$$q = q_{\rm t}/y^2,\ \ q_{\rm t} = x_{\rm t}\,x_{1\rm t}w,$$
$$Q = Q_{\rm t}/y,\ \ Q_{\rm t}^2 = (x_{\rm t} - x_{1\rm t})^2 + 2\,q_{\rm t},\ \ Q_* \approx (Q_{\rm t}^2 + 2r\,q_{\rm t})/y^2. \qquad (9.2.1)$$

In the limiting case of small frequencies, one must distinguish the subcases of different relations between the typical energy that photon loses (or gains) in the interaction and the typical frequency scale where n changes essentially. If we investigate the broadening of an atomic spectral line in the optical or ultraviolet region by non-relativistic electrons, then the line absorption coefficient profile is narrower than the mean photon displacement in frequencies due to the electron scattering. Then we can neglect recoil effect and put $x_{1\rm t} = x_{\rm t} = x_0$ everywhere except difference $x_{1\rm t} - x_{\rm t}$. Rejecting corrections of order $1/y$, we find RF that reflects only the Doppler effect:

$$S(x, x_1, \mu){\rm d}x_1 \sim \frac{3(1+\mu^2)\sqrt{y}}{32\pi^{3/2}x_0\sqrt{w}}\exp\left(-\frac{y(x_{\rm t} - x_{1\rm t})^2}{4wx_0^2}\right){\rm d}x_{1\rm t}. \qquad (9.2.2)$$

This RF was deduced by Hummer & Mihalas (1967) (see also Mihalas, 1978).

On the other hand, if we calculate the continuous spectrum in the region wider than the photon displacement due to the scattering, then one can transpose the integral kinetic equation to the differential (on frequency) equation. Substituting (9.2.1) in (9.1.10), expanding all functions in $1/y$ and taking into account only the terms of order $1/y$, one obtains the expression for RF in that approximation:

$$S_{\rm t}(x_{\rm t}, x_{1\rm t}, \mu) = S(x, x_1, \mu)/y = \frac{3}{16\,\pi}\left\{\frac{w(1+\mu^2)x_{\rm t}}{y}\left[x_{\rm t}\delta^{''}(x_{1\rm t}-x_{\rm t})\right.\right. \tag{9.2.3}$$
$$- \quad \left.(x_{\rm t}+2)\delta^{'}(x_{1\rm t}-x_{\rm t})\right] + \left.\left[(1+\mu^2)\left(1+\frac{w\,x_{\rm t}}{y}\right) - \frac{2\,\mu}{y}(1+4\,\mu-3\,\mu^2)\right]\delta(x_{1\rm t}-x_{\rm t})\right\}.$$

If the radiation field is isotropic and homogeneous, it is sufficient to use RF averaged over the directions (i.e. over μ). From (9.2.3) we can obtain

$$\begin{aligned} S_{\rm t}(x_{\rm t}, x_{1\rm t}) &= \{[1+(x_{\rm t}-2)/y]\delta(x_{1\rm t}-x_{\rm t}) \\ &+ \left. x_{\rm t}[x_{\rm t}\delta^{''}(x_{1\rm t}-x_{\rm t}) - (x_{\rm t}+2)\delta^{'}(x_{1\rm t}-x_{\rm t})]\right\}/4\pi. \end{aligned} \tag{9.2.4}$$

Using the representation of RF in the form (9.2.3), one can deduce from (8.4.1) the equation, which was obtained by Babuel-Peyrissac & Rouvillois (1969):

$$\begin{aligned} &\frac{1}{c}\frac{\partial n}{\partial t} + \boldsymbol{\omega}\cdot\boldsymbol{\nabla} n = \sigma_{\rm T}\, n_{\rm e}\frac{3}{16\pi}\int {\rm d}^2\omega_1 \\ &\times \quad \left\{(n_1-n)\left[(1+\mu^2)\left(1-2\,w\frac{x_{\rm t}}{y}\right)+\frac{2}{y}(1-2\,\mu-3\,\mu^2+2\,\mu^3)\right]\right. \\ &+ \quad \left. w\,(1+\mu^2)\frac{x_{\rm t}}{y}\left[x_{\rm t}\left(n_1^{''}+n_1^{'}(1+2\,n)\right)+4\left(n_1^{'}+n_1(1+n)\right)\right]\right\}. \end{aligned} \tag{9.2.5}$$

Here $n = n(x, \boldsymbol{\omega})$ and $n_1 = n(x, \boldsymbol{\omega}_1)$. The dependence on coordinates and time is omitted.

The presentation (9.2.4) leads us to the Kompaneets equation (Kompaneets, 1956):

$$\frac{1}{c}\frac{\partial n}{\partial t} = \frac{\sigma_{\rm T}\, n_{\rm e}}{y}\frac{1}{x_{\rm t}^2}\frac{\partial}{\partial x_{\rm t}}\left[x_{\rm t}^4\left(n+n^2+\frac{\partial n}{\partial x_{\rm t}}\right)\right], \tag{9.2.6}$$

where $n = n(x, t)$ depends only on frequency and time.

10 Conclusions

We have considered in this review, we hope, almost all known analytical results concerning single Compton scattering of polarized light. Let us list them briefly.

The kinetic equation describing this scattering is formulated. There is no limitations on the photon energy. It is assumed that the electron gas is non-degenerate and the distribution function is known. The equation is written in terms of the polarization density matrix and of the Stokes parameters, which are dimensionless relativistically invariant quantities.

The terms of the equation describing the extinction and the scattering of radiation are considered successively. The mean coefficient of extinction due to Compton scattering and the mean values of the powers (the first and the second) of frequency of the scattered photon are defined. The averaging is performed in three stages: first, over the direction of the scattered photon; second, over the direction of the initial electron; and, finally, over the electron distribution, which is assumed to be relativistic Maxwellian.

The quantities which have a direct physical sense, such as the energy balance between the electron and photon gases, the dispersion of the frequency of the scattered photon and the radiation pressure on the electron gas, are considered.

Together with the exact formulae for all quantities considered, we have given the asymptotic and approximate formulae in different limiting cases: weakly relativistic and ultrarelativistic electrons with fixed energy, soft photons (with energy less than the electron rest energy), and Maxwellian electron gas of low and high temperature.

The scattering matrix is obtained for the case when the electron distribution in some frame of reference is isotropic. It is shown that the scattering matrix consists of three co-multipliers. Two of them are the same as for the Thomson (Rayleigh) scattering (rotation matrices), and the central matrix has five functions describing the redistribution in frequency and direction. These five dimensionless RFs depend on the photon frequencies before and after scattering and the scattering angle.

The expansions of RF in a series or asymptotic representation are obtained for the same limiting cases as for the mean values. The RF of intensity is investigated in detail. Averaging over direction (over the cosine of the scattering angle) with the weights 1 and μ is performed. The relations between RF and the mean values are found. The zeroth and the first moments of the kinetic equation are deduced. The approximate expressions of RF for nearly non-relativistic electrons and small photon frequencies are obtained. With the aid of these expressions it is easy to derive the known Fokker-Planck equations deduced earlier in a different way.

All exact equations and relations are relativistically covariant. Calculations are performed and results are given in the frame of reference co-moving with the electron gas. However, using the Lorentz transformations, it is not difficult to obtain these results in the covariant form.

The equations which take into account the polarization of radiation are linear. The form of the equation describing the change of intensity for the induced scattering is known. But the equation, describing the non-linear processes due to the scattering of polarized light is not yet formulated and deducting it is the problem. First attempts in that direction have been made — namely Stark (1981) has considered the induced scattering for the linear polarization in the Fokker-Planck approximation.

Now the question arises as to how to solve the relativistic kinetic equations, i.e. to calculate the multiple scattering. Some approximate methods for the special cases were mentioned in the Introduction. In the general case this problem remains unresolved. The solution of this problem can extend possible applications of the theory to the interpretation and prediction of the observation of relativistic astrophysical objects. Some applications of the developed formalism of the scattering matrix were made by Poutanen & Vilhu (1993) for a two-phase model of accretion disc.

It is necessary to note that for the application of the equations considered to the case of scattering of gamma-photons, we must take into account the possibility of the electron-positron pair production and the subsequent annihilation. Significant advancement in that direction was made in the works by Svensson (1987), Zdziarski et al. (1990), Zdziarski & Coppi (1991). We hope that the solutions of the problems outlined are not far distant.

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