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Relativistic Kinetic Equation for Induced Compton Scattering of Polarized Radiation

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Abstract. The relativistic kinetic equations describing time evolution and space dependence of the density matrices of polarized photons and electrons interacting via Compton scattering are deduced from the quantum Liouville equation. The induced scattering and exclusion principle are taken into account. The Bogoliubov method is used in the frame of quantum electrodynamics. The equation for polarized radiation scattered by unpolarized electrons is considered as a particular case and is reformulated in terms of the Stokes parameters. The expressions for the scattering amplitudes and cross-sections are derived simultaneously.

Key words. methods: analytical – radiation mechanisms: general – plasmas – polarization – scattering

1. Introduction

Compton scattering is an important physical process in many astrophysical systems, such as active galactic nuclei, X-ray binaries, and pulsar magnetospheres (see e.g. reviews by Blandford & Scharlemann 1975; Pozdnyakov, Sobol', & Sunyaev 1983; Poutanen 1998). Inverse Compton scattering (i.e. scattering of soft photons by hot electron gas) is believed to be the main mechanism of the X-ray/gamma-ray production in accreting X-ray binaries (e.g. Sunyaev & Titarchuk 1980; Poutanen & Svensson 1996), gamma-ray bright active galactic nuclei (e.g. Sikora et al. 1997), and possibly gamma-ray bursts (Stern 1999; Ghisellini & Celotti 1999). The hard X-rays can interact with the cold material via classical Compton scattering losing their energy and producing a cutoff in the "Compton reflected" spectrum (e.g. George & Fabian 1991; Poutanen, Nagendra, & Svensson 1996). Induced (stimulated) Compton scattering becomes very important when the radiation brightness temperature is large $k_{\rm B}T_{\rm b}/mc^2 \gg 1/\tau_{\rm T}$, where $\tau_{\rm T}$ is Thomson optical thickness. Induced effects can distort the low frequency part of the radio-source spectra even when $\tau_{\rm T}$ is small (Sunyaev 1971) and can influence the heating of electrons near active galactic nuclei and pulsars (Levich & Sunyaev 1971).

The process of multiple scattering of radiation is described by a kinetic equation. This equation can be easily written if we neglect the induced scattering and/or the polarization (see, e.g. Nagirner & Poutanen 1994 and references therein). Induced scattering leads to the appearance of nonlinear terms in the kinetic equation. Since the polarization of radiation is described by four parameters, one must write a vector kinetic equation, i.e. a system of four equations. The kinetic equation combining these two effects (polarization and induced scattering) was not derived self-consistently up to now. The aim of the present paper is to fill up this gap.

The first kinetic equation for Compton scattering with induced scattering was written by Kompaneets (1956) and is known under his name. He considered multiple scattering of homogeneous and isotropic radiation in infinite space filled with homogeneous, nondegenerate thermal electron gas. The gas was assumed to be non-relativistic ($k_{\rm B}T_{\rm e} \ll mc^2$), the radiation to be rather soft ($h\nu \ll mc^2$), and the intensity of radiation to be a sufficiently smooth function of frequency. Because of small changes of photon frequency in a single scattering, the scattering integral was transformed to a differential operator by the Fokker—Planck method. The equation was rediscovered by Weymann (1966). More general Fokker—Planck equations were deduced by Cooper (1971) and Barbosa (1982) for more energetic electrons and photons, and by Molodtsov (1994) for the anisotropic, degenerate and moving electrons. For very cold electron gas, the Fokker—Planck equation was derived by Ross et al. (1978).

The induced scattering effects strongly depend on the angular distribution of the radiation field. Babuel-Peyrissac & Rouvillois (1969) generalized the Kompaneets equation to nonhomogeneous and anisotropic radiation. Their equation contains an integral operator in angular variables, but a differential operator in frequency. A simplified form of that equation was used by a number of authors to study induced effects in pulsar magnetospheres and radio sources in active galactic nuclei (Wilson & Rees 1978; Coppi, Blandford, & Rees 1993; Sincell & Krolik 1994; Lyubarskii & Petrova 1996; Sincell & Coppi 1996).

As it is mentioned above, the induced scattering makes equations nonlinear. This effect can be simply taken into account in the equation if we limit ourselves to one parameter, intensity I, ignoring polarization. The scattering rate is increased by stimulated scattering by a factor $1 + h^2 I(\boldsymbol{k})/2ck^3$ (where $\boldsymbol{k}$ is the photon momentum after scattering) comparing with the rate when only spontaneous scattering is considered. It is this factor that the authors of the mentioned works (Kompaneets 1956; Babuel-Peyrissac & Rouvillois 1969; Molodtsov 1994) have introduced into the equations. Often kinetic equations (e.g. radiative transfer equation) are deduced heuristically from intuitive considerations and previous experience. However, in the case when polarization including induced effects is considered, our intuition and experience are not enough and the phenomenological approach fails. Therefore, one has to resort to deductive methods. Note that in nearly all the works on Compton scattering the electrons were assumed to be unpolarized and isotropic (or monodirectional).

The first attempt to find out a rule for consideration of induced processes simultaneously with polarization was undertaken by Pomraning (1974). In terms of plane electromagnetic waves, he considered a source of polarized emission interacting with already existing radiation field and deduced a matrix describing the transformation of the Stokes parameters. The Stokes vector corresponding to the induced process was presented as a product of a 4×4 matrix $\mathbf{N}$ with the Stokes vector of the incoming radiation. Calculating the elements of the matrix $\mathbf{N}$, he averaged the product of the quantities corresponding to the emitted (source photons) and passing by (field photons) waves over the distributions of their phases assuming that no correlation exists between the source and the field. This result is not applicable to scattering because the scattered radiation does not have random phases. In the resulting expression for $\mathbf{N}$, only the elements in the first row and the first column (i.e. related to the intensity) are correct in the case of scattering (Stark 1981).

Derivation of the kinetic equation for the induced scattering of polarized radiation by non-relativistic electrons was given by Wilson (1978). He used the Maxwell equations, expansion electromagnetic field on set of harmonic oscillators, second quantization method and the non-relativistic perturbation theory. The scattering (a process of the fourth order in the approximation used) was represented as consequent absorption and emission. The products of the probabilities of these processes were replaced by the elements of the scattering matrix deduced according to the rules of quantum electrodynamics. While such a deduction is not self-consistent, the resulting equation is correct.

Wilson's results were used by Stark (1981) to deduce the generalization of the Babuel-Peyrissac—Rouvillois (1969) equation for stimulated scattering to linearly polarized radiation (two Stokes parameters). Hansen & Lilje (1999) have corrected inaccuracies and misprints in Stark's work. Using a version of Wilson's equation containing only non-linear terms, Wilson (1982) and Coppi et al. (1993) showed that the induced scattering can enhance the polarization (comparing to the spontaneous scattering) producing also its strong frequency dependence. Induced Compton backscattering, for example, can amplify an incident ~ 1 per cent polarization up to ~ 50 per cent (Sincell & Coppi 1996). Accounting for the polarization of radiation may lead to the results qualitatively different from those obtained neglecting polarization. A small change in the scattering rate (affected by polarization) can be strongly amplified due to the non-linearity of the process.

The relativistic kinetic equation for the Stokes parameters taking into account the induced scattering was presented by Nagirner (1994) without deduction. The work of Wilson was not accessible for the author at that time. Shortly after that, the work of Ioffe made at the beginning of the 1950-ies was published (Ioffe 1994). In this work the equation was deduced (as Ioffe pointed out, with an approximate and a simplified method) for the so called polarization tensor which differs from the density matrix because the unphysical scalar and longitudinal photons are not excluded.

In this paper, we derive the relativistic kinetic equation for the photon gas interacting with the electron gas. The way of deduction is based on the methods of Bogoliubov (Bogoliubov & Gurov 1947) and Baranger (1958). The scheme closest to ours is used in Silin (1971), where the non-relativistic kinetic equation for interacting electrons was derived. In our paper, at every step of the derivation all the expressions and relations are demonstrated to be either explicitly relativistically covariant or they can be changed to relativistic forms. The final equations are explicitly relativistic. In order to elucidate these facts and to introduce suitable for our purposes notations, we are urged to present a number of known relations and schemes. We thus fully describe the method of the derivation of the equations in a self-contained way.

We make assumptions usual for the kinetic theory. Only binary collisions are accounted for. We assume the molecular chaos, i.e. the gas states are characterized by one-particle distribution functions. The characteristic temporal and space scales of a single interaction are assumed to be much smaller than the scales of significant changing of radiation and electron gas states. We use the principle of weakening of the correlations, which says that the correlations between particle states are expressed in terms of the same one-particle distribution functions. There are no other limitations on the states of interacting particles. Simultaneously with the kinetic equations, we obtain the expressions for the scattering amplitudes and (well known) cross-sections.

We first assume that both electrons and photons are polarized, then we average the equations over electron polarizations. An arbitrary anisotropy and nonhomogeneity of the radiation field and electron distribution are allowed. We also derive the kinetic equation describing the evolution of the electron gas.

2. Density Matrix of the Photon-Electron Gas

2.1. Operators of Creation and Annihilation of Photons

According to the method of second quantization (see e.g. Bogoliubov & Shirkov 1959; Schweber 1961), the two types of vector-operators are introduced satisfying the following commutation relations

$$[a_\mu(\boldsymbol{k}),\overline{a}_\nu(\boldsymbol{k}')]=a_\mu(\boldsymbol{k})\overline{a}_\nu(\boldsymbol{k}')-\overline{a}_\nu(\boldsymbol{k}')a_\mu(\boldsymbol{k})=-g_{\mu\nu}k\delta(\boldsymbol{k}-\boldsymbol{k}'),\quad \mu,\,\nu=0,1,2,3, \tag{1}$$

where $\boldsymbol{k}$ is the photon momentum, $k=|\boldsymbol{k}|$ and the metrics $\{g_{\mu\nu}\}=\mathrm{diag}\{1,-1,-1,-1\}$. The commutators of all other components are equal to zero, i.e. $[a_\mu(\boldsymbol{k}),a_\nu(\boldsymbol{k}')]=[\overline{a}_\mu(\boldsymbol{k}),\overline{a}_\nu(\boldsymbol{k}')]=0$. The commutator of the zeroth components of the vector-operator, $a_0(\boldsymbol{k})$ and $\overline{a}_0(\boldsymbol{k}')$, in equation (1) differs from others. Operators a and $\overline{a}$ are interpreted as the operators of annihilation and creation of a photon, respectively. Their product $\overline{a}_{(\lambda)}(\boldsymbol{k})a_{(\lambda)}(\boldsymbol{k})$ (no summation) is the operator of the number of photons of momentum $\boldsymbol{k}$ and polarization λ.

Generally speaking, one has to quantize the photon field in a finite volume V (box), for example, of the parallelepiped shape with the sides equal to $L_\mathrm{x}, L_\mathrm{y}, L_\mathrm{z}$, $V=L_\mathrm{x}\ L_\mathrm{y}\ L_\mathrm{z}$. Then in every dimension of this volume there can be a countable number of standing waves of the form $(2\pi\hbar)^{-3/2}\exp(i\boldsymbol{k}\boldsymbol{r}/\hbar)$, where $\boldsymbol{k}=2\pi\hbar(n_\mathrm{x}/L_\mathrm{x},n_\mathrm{y}/L_\mathrm{y},n_\mathrm{z}/L_\mathrm{z})$ and $n_\mathrm{x},n_\mathrm{y},n_\mathrm{z}$ are integers. Correspondently, the number of creation and annihilation operators should be countable and the commutation rules should contain discrete δ-functions, i.e. the Kronecker symbols. However, in the limit $L_\mathrm{x},L_\mathrm{y},L_\mathrm{z}\to\infty$ the countable set of standing waves transforms to the continuum and all relations for the finite volume and its limiting case can be written in a unified way (Bogoliubov & Shirkov 1959).

For example, the orthogonality condition for the discrete case with the integration over the box has the same form as in the continuum case, if the δ-function and the volume element of the three-dimensional momentum space are taken in the discrete form:

$$\frac{1}{(2\pi\hbar)^3}\int\exp\{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{r}/\hbar\}\mathrm{d}^3r=\delta(\boldsymbol{k}-\boldsymbol{k}'),\quad \delta(\boldsymbol{k}-\boldsymbol{k}')=\frac{V}{(2\pi\hbar)^3}\delta_{n_\mathrm{x}n'_\mathrm{x}}\delta_{n_\mathrm{y}n'_\mathrm{y}}\delta_{n_\mathrm{z}n'_\mathrm{z}},\quad \mathrm{d}^3k=\frac{(2\pi\hbar)^3}{V}. \tag{2}$$

Therefore, we keep nearly all the quantities and relations in the form corresponding to the continuum, while they are valid for the discrete case as well (i.e., for a finite box).

Let us expand vectors $\underline{a}(\boldsymbol{k})=\{a_0(\boldsymbol{k}),\boldsymbol{a}(\boldsymbol{k})\}=\{a^\mu(\boldsymbol{k})\}$ and $\underline{\overline{a}}(\boldsymbol{k})$ along the unit vectors of the four-dimensional basis whose vectors are orthonormal and their system is full:

$$\underline{a}(\boldsymbol{k})=a^{(\lambda)}(\boldsymbol{k})\underline{e}_{(\lambda)},\quad \underline{\overline{a}}(\boldsymbol{k})=\overline{a}^{(\lambda)}(\boldsymbol{k})\underline{e}_{(\lambda)},\quad \underline{e}_{(\lambda)}\underline{e}_{(\lambda')}=e_{(\lambda)\mu}e^\mu_{(\lambda')}=g_{\lambda\lambda'},\quad e_{(\lambda)\mu}e^{(\lambda)}_\nu=g_{\mu\nu} \tag{3}$$

(hereafter, summation over repeated indices is assumed). Raising and lowering the indices in brackets numbering the unit vectors are made in the same way as for the case of ordinary indices numbering the vector components. The coefficients of the expansion, $a^{(\lambda)}(\boldsymbol{k})$ and $\overline{a}^{(\lambda)}(\boldsymbol{k})$, are Lorentz invariants (i.e., they are scalars). They satisfy the commutation relations which follow from equations (1) and (3)

$$[a^{(\lambda)}(\boldsymbol{k}),\overline{a}^{(\lambda')}(\boldsymbol{k}')]=-g^{\lambda\lambda'}k\delta(\boldsymbol{k}-\boldsymbol{k}'),\quad \lambda,\lambda'=0,1,2,3. \tag{4}$$

In order for the selected basis to become a polarization basis, their unit vectors should satisfy the additional relations that the two unit vectors corresponding to the transverse polarization should be orthogonal to the photon momentum

$$\underline{k}\,\underline{e}_{(1)}(\boldsymbol{k})=\underline{k}\,\underline{e}_{(2)}(\boldsymbol{k})=0 \tag{5}$$

where $\underline{k}=\{k,\boldsymbol{k}\}$ is the photon four-momentum and the scalar product $\underline{k}\,\underline{e}=k^\mu e_\mu=ke_0-\boldsymbol{k}\cdot\boldsymbol{e}$. As a result of such orthogonality, the photon momentum can be represented as a linear combination of the two unit vectors of the polarization basis

$$\underline{k}=[\underline{k}\,\underline{e}_{(0)}(\boldsymbol{k})][\underline{e}_{(0)}(\boldsymbol{k})+\underline{e}_{(3)}(\boldsymbol{k})]. \tag{6}$$

Then the following equality holds

$$\underline{k}\,\underline{a}(\boldsymbol{k})=[\underline{k}\,\underline{e}_{(0)}(\boldsymbol{k})][a^{(0)}(\boldsymbol{k})-a^{(3)}(\boldsymbol{k})], \tag{7}$$

and similarly for the conjugate operator. Obviously, the unit vectors of the polarization basis are not arbitrary but related to the vector of the photon momentum $\boldsymbol{k}$. However, the commutation relations (4) are still valid.

In classical electrodynamics, the transversality relation was given by equality to zero of the quantity (not an operator) formally coinciding with (7). The equality of the amplitudes of scalar and longitudinal potentials (or equality of both to zero) was the consequence of that relation. In quantum electrodynamics, this relation is too stiff. One cannot require equality of $a^{(0)}(\boldsymbol{k})$ and $a^{(3)}(\boldsymbol{k})$, since they satisfy different commutation relations which easily can be seen from relation (4). The solution of that problem was given by Gupta (1950) and Bleuler (1950) (see also Bogoliubov & Shirkov 1959; Schweber 1961). We use their method.

Only the operators with indices 1 and 2 are physical and only those should enter the quantities having physical meaning. These operators satisfy the commutation relations

$$[a^{(s)}(\boldsymbol{k}),\overline{a}^{(s')}(\boldsymbol{k}')] = [a_{(s)}(\boldsymbol{k}),\overline{a}_{(s')}(\boldsymbol{k}')] = \delta_{ss'}k\delta(\boldsymbol{k}-\boldsymbol{k}'), \quad s,s'=1,2. \tag{8}$$

2.2. Photon States

The states of the electromagnetic field in the second-quantized representation are described by the vectors with the arguments equal to given momenta and polarization projections of the definite number of photons. The initial vector that is used to obtain all other vectors is the state vector of the photon vacuum Ψ_0. It is normalized and satisfies the condition of absence of photons: $\overline{\Psi}_0\Psi_0 = 1$, $a^{(\lambda)}(\boldsymbol{k})\Psi_0 = 0$. The second equality is valid for any momentum $\boldsymbol{k}$ and any polarization λ.

All other photon state vectors can be obtained from the vacuum state vector by applying the creation operators but the vectors containing information about the non-physical scalar and longitudinal photons should be excluded. In the physical states, the operators of photon creation of these photons should appear only in the combination $\overline{a}^{(0)}(\boldsymbol{k}) + \overline{a}^{(3)}(\boldsymbol{k})$, so that there always exist equal amount of scalar and longitudinal photons. With the action of the operator $a^{(0)}(\boldsymbol{k}) - a^{(3)}(\boldsymbol{k})$ on to such states, such photons mutually cancel out. The product of the operators of creation and annihilation in such a combination gives $\overline{a}^{(0)}(\boldsymbol{k})a^{(0)}(\boldsymbol{k}) - \overline{a}^{(3)}(\boldsymbol{k})a^{(3)}(\boldsymbol{k})$, which acting on to the vector of the physical states gives zero. The full product of the operators of creation and annihilation is reduced to $\underline{\overline{a}}(\boldsymbol{k})\underline{a}(\boldsymbol{k}) = -\overline{a}_{(s)}(\boldsymbol{k})a_{(s)}(\boldsymbol{k})$. Here the summation is done over $s = 1,2$ (in the several following sections we use only lower indices s). We omit unphysical photons hereafter from our considerations.

The physically allowed states of N photons with fixed momenta and polarizations can be represented in the form

$$\Psi_{s_1\ldots s_N}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N) = \frac{1}{\sqrt{N!}}\overline{a}_{(s_1)}(\boldsymbol{k}_1)\ldots\overline{a}_{(s_N)}(\boldsymbol{k}_N)\Psi_0. \tag{9}$$

The related arguments s and $\boldsymbol{k}$ can be moved simultaneously and rearranged in any order, since operators commutate.

The orthogonality condition for the vectors of physical states follows from the commutation relations (8) and takes the form

$$\overline{\Psi}_{s'_1\ldots s'_N}(\boldsymbol{k}'_1,\ldots,\boldsymbol{k}'_N)\Psi_{s_1\ldots s_N}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N) = \frac{1}{N!}\sum_{\alpha_1,\ldots,\alpha_N}\prod_{j=1}^{N}[k_j\delta_{s_j s'_{\alpha_j}}\delta(\boldsymbol{k}_j-\boldsymbol{k}'_{\alpha_j})], \tag{10}$$

where the sum is taken over all permutations $\alpha_1,\ldots\alpha_N$ of indices $1,2,\ldots,N$. These indices can alternatively be assigned to the arguments without primes, which is equivalent to the change from the sum over rows to the sum over columns.

The results of the action of the creation and annihilation operators on to the state vector are given by

$$a_{(s)}(\boldsymbol{k})\Psi_{s_1\ldots s_N}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N) = \frac{1}{\sqrt{N}}\sum_{r=1}^{N}[k\delta_{ss_r}\delta(\boldsymbol{k}-\boldsymbol{k}_r)]\ \Psi_{s_1\ldots s_{r-1}s_{r+1}\ldots s_N}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_{r-1},\boldsymbol{k}_{r+1},\ldots,\boldsymbol{k}_N), \tag{11}$$

$$\overline{a}_{(s)}(\boldsymbol{k})\Psi_{s_1\ldots s_N}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N) = \sqrt{N+1}\ \Psi_{ss_1\ldots s_N}(\boldsymbol{k},\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N). \tag{12}$$

The product of these operators acts on to the vector as follows

$$\overline{a}_{(s)}(\boldsymbol{k})a_{(s')}(\boldsymbol{k}')\Psi_{s_1\ldots s_N}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N) = \sum_{r=1}^{N}k'\delta_{s's_r}\delta(\boldsymbol{k}'-\boldsymbol{k}_r)\Psi_{s_1\ldots s_{r-1}ss_{r+1}\ldots s_N}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_{r-1},\boldsymbol{k},\boldsymbol{k}_{r+1},\ldots,\boldsymbol{k}_N). \tag{13}$$

Arbitrary physical state with N transverse photons can be described by the vector

$$\Psi_N = \int\frac{\mathrm{d}^3k_1}{k_1}\ldots\frac{\mathrm{d}^3k_N}{k_N}c_{s_1\ldots s_N}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)\Psi_{s_1\ldots s_N}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N), \tag{14}$$

where the sum is taken over the repeated indices $s_j = 1,2$. The coefficients of the expansion can be found as usual Fourier coefficients. We assume that vectors of the physical state (14) are dimensionless. They are normalized as $\overline{\Psi}_N\Psi_N = 1$. In order to prove that, the orthogonality condition (10) should be used and the integral over momenta $\boldsymbol{k}'_i$ must be taken applying the permutation property for indicies and momenta.

2.3. Quantization of the Electron-Positron Field

In the second-quantization of the electron and positron fields (as in the case of the electromagnetic field), the operators of annihilation and creation satisfy the relations of the form

$$\{b_\tau(\boldsymbol{p}), b^\dagger_{\tau'}(\boldsymbol{p}')\} = b_\tau(\boldsymbol{p})b^\dagger_{\tau'}(\boldsymbol{p}') + b^\dagger_{\tau'}(\boldsymbol{p}')b_\tau(\boldsymbol{p}) = \delta_{\tau\tau'}p_0\delta(\boldsymbol{p}-\boldsymbol{p}'), \text{ for electrons,} \tag{15}$$

and the same for the positron operators $d_\sigma(\boldsymbol{p})$. Relation (15) (and a similar one for positrons) contains anticommutators (not commutators), which is a consequence of the fact that these particles are fermions and satisfy the Pauli exclusion principle. All other binary combinations of the operators anticommutate.

The electron-positron vacuum is defined with normalized vector Φ_0 as a state where there are no particles, i.e.

$$b_\tau(\boldsymbol{p})\Phi_0 = 0, \quad d_\sigma(\boldsymbol{p})\Phi_0 = 0, \quad \Phi_0^\dagger\Phi_0 = 1. \tag{16}$$

Then the state with N_- electrons can be defined via vector

$$\Phi_{\tau_1...\tau_{N_-}}(\boldsymbol{p}_1, ..., \boldsymbol{p}_{N_-}) = \frac{1}{\sqrt{N_-!}} b^\dagger_{\tau_1}(\boldsymbol{p}_1)...b^\dagger_{\tau_{N_-}}(\boldsymbol{p}_{N_-})\Phi_0. \tag{17}$$

The order of the indices and corresponding momenta can be changed arbitrarily, but since the operators now anticommutate, the sign of the vector changes depending whether a permutation is odd or even. The positron states can be defined in a similar manner. Since the scattering of photons by positrons does not substantially differ from the electron case, we can assume that it is only electrons that interact with photons and we do not introduce positron states.

The operators of creation and annihilation act on to the state vectors in the following way

$$b_\tau(\boldsymbol{p})\Phi_{\tau_1...\tau_{N_-}}(\boldsymbol{p}_1, ..., \boldsymbol{p}_{N_-}) = \frac{1}{\sqrt{N_-}} \sum_{j=1}^{N_-} (-1)^{j-1}\delta_{\tau\tau_j}p_0\delta(\boldsymbol{p}-\boldsymbol{p}_j)\Phi_{\tau_1...\tau_{j-1}\tau_{j+1}...\tau_{N_-}}(\boldsymbol{p}_1, ..., \boldsymbol{p}_{j-1}\boldsymbol{p}_{j+1}, ..., \boldsymbol{p}_{N_-}), \tag{18}$$

$$b^\dagger_\tau(\boldsymbol{p})\Phi_{\tau_1...\tau_{N_-}}(\boldsymbol{p}_1, ..., \boldsymbol{p}_{N_-}) = \sqrt{N_- + 1}\ \Phi_{\tau\tau_1...\tau_{N_-}}(\boldsymbol{p}, \boldsymbol{p}_1, ..., \boldsymbol{p}_{N_-}). \tag{19}$$

The products of operators give the following

$$b^\dagger_\tau(\boldsymbol{p})b_{\tau'}(\boldsymbol{p}')\Phi_{\tau_1...\tau_{N_-}}(\boldsymbol{p}_1, ..., \boldsymbol{p}_{N_-}) = \sum_{j=1}^{N_-} \delta_{\tau'\tau_j}p'_0\delta(\boldsymbol{p}'-\boldsymbol{p}_j)\Phi_{\tau_1...\tau_{j-1}\tau\tau_{j+1}...\tau_{N_-}}(\boldsymbol{p}_1, ..., \boldsymbol{p}_{j-1}, \boldsymbol{p}, \boldsymbol{p}_{j+1}, ..., \boldsymbol{p}_{N_-}). \tag{20}$$

Normalization of the state vectors is defined in a manner similar to photons. One has to change the order of the creation and annihilation operators many times in order for the former to be in the beginning and for the later to be in the end. The only difference is that a minus sign appears for odd permutation. The result can be written in the determinant form. For electrons,

$$\Phi^\dagger_{\tau'_1...\tau'_{N_-}}(\boldsymbol{p}'_1, ..., \boldsymbol{p}'_{N_-})\Phi_{\tau_1...\tau_{N_-}}(\boldsymbol{p}_1, ..., \boldsymbol{p}_{N_-}) = \frac{1}{N_-!}\det\left(\delta_{\tau_i\tau'_j}p_{i0}\delta(\boldsymbol{p}_i - \boldsymbol{p}'_j)\right). \tag{21}$$

The vector of an arbitrary state with N_- electrons can be represented via an expansion over the elementary state vectors

$$\Phi_{N_-} = \int \frac{\mathrm{d}^3p_1}{p_{10}}...\frac{\mathrm{d}^3p_{N_-}}{p_{N_-0}} c_{\tau_1...\tau_{N_-}}(\boldsymbol{p}_1, ..., \boldsymbol{p}_{N_-})\Phi_{\tau_1...\tau_{N_-}}(\boldsymbol{p}_1, ..., \boldsymbol{p}_{N_-}). \tag{22}$$

The interchange of the arguments of the coefficients c changes their sign. The normalization of the state vectors, $\Phi^\dagger_{N_-}\Phi_{N_-} = 1$, can be easily obtained from equations (21) and (22).

2.4. Density Matrices for Photons and Electrons

The density matrix of N-photon state (N is fixed) can be defined as an averaged dyad product of the state vector with its conjugate

$$\begin{aligned}\rho_N &= \langle\Psi_N\overline{\Psi}_N\rangle \\ &= \int \frac{\mathrm{d}^3k'_1}{k'_1}...\frac{\mathrm{d}^3k'_N}{k'_N}\frac{\mathrm{d}^3k_1}{k_1}...\frac{\mathrm{d}^3k_N}{k_N}\langle c^*_{s'_1...s'_N}(\boldsymbol{k}'_1, ..., \boldsymbol{k}'_N)c_{s_1...s_N}(\boldsymbol{k}_1, ..., \boldsymbol{k}_N)\rangle\Psi_{s_1...s_N}(\boldsymbol{k}_1, ..., \boldsymbol{k}_N)\overline{\Psi}_{s'_1...s'_N}(\boldsymbol{k}'_1, ..., \boldsymbol{k}'_N).\end{aligned} \tag{23}$$

This product is an operator, since the elementary state vectors contain the creation and annihilation operators.

Let us introduce the notation for the kernel of the density matrix operator

$$\rho^{s'_1...s'_N}_{s_1...s_N}\begin{pmatrix}\boldsymbol{k}'_1 \,...\, \boldsymbol{k}'_N\\ \boldsymbol{k}_1 \,...\, \boldsymbol{k}_N\end{pmatrix} = N!\,\langle c^*_{s'_1...s'_N}(\boldsymbol{k}'_1,...,\boldsymbol{k}'_N)c_{s_1...s_N}(\boldsymbol{k}_1,...,\boldsymbol{k}_N)\rangle. \tag{24}$$

This kernel, being the N-particle photon distribution function in the momentum representation, cannot be represented as a product after averaging of the expansion coefficients c. The density matrix is then written in the form

$$\rho_N = \frac{1}{N!}\int\frac{\mathrm{d}^3k'_1}{k'_1}...\frac{\mathrm{d}^3k'_N}{k'_N}\frac{\mathrm{d}^3k_1}{k_1}...\frac{\mathrm{d}^3k_N}{k_N}\rho^{s'_1...s'_N}_{s_1...s_N}\begin{pmatrix}\boldsymbol{k}'_1 \,...\, \boldsymbol{k}'_N\\ \boldsymbol{k}_1 \,...\, \boldsymbol{k}_N\end{pmatrix}\Psi_{s_1...s_N}(\boldsymbol{k}_1,...,\boldsymbol{k}_N)\overline{\Psi}_{s'_1...s'_N}(\boldsymbol{k}'_1,...,\boldsymbol{k}'_N), \tag{25}$$

and the kernel can be represented through the operator

$$\rho^{s'_1...s'_N}_{s_1...s_N}\begin{pmatrix}\boldsymbol{k}'_1 \,...\, \boldsymbol{k}'_N\\ \boldsymbol{k}_1 \,...\, \boldsymbol{k}_N\end{pmatrix} = N!\,\overline{\Psi}_{s_1...s_N}(\boldsymbol{k}_1,...,\boldsymbol{k}_N)\rho_N\Psi_{s'_1...s'_N}(\boldsymbol{k}'_1,...,\boldsymbol{k}'_N). \tag{26}$$

Only diagonal elements (i.e., elements with the same primed and non-primed indices and arguments) are real and non-negative.

It is obvious from definition (23), that the density matrix is a self-conjugate operator, and it follows from (24) and (25) that its kernel is a self-conjugate matrix, i.e. its Hermitian conjugation (complex conjugation and the replacement of the lower and the upper arguments with each other) is equal to the same kernel. The trace of the density matrix $\mathrm{Sp}\,\rho_N = \langle\overline{\Psi}_N\Psi_N\rangle = 1$ by virtue of the normalization of the state vector.

The density matrix carries a lot of information, much more than the distribution function or even the polarization matrix which can be expressed through the density matrix. However, in order to derive the kinetic equation, the distribution functions are not enough and one has to introduce a set of functions containing the groups of variables with the dimension less than N, but larger than 1. We call them truncated density matrices.

Together with the N-particle density matrix we introduce matrices whose kernels can be expressed through the integrals of the kernel of the original matrix. A kernel of order l

$$\rho^{s'_1...s'_l}_{s_1...s_l}\begin{pmatrix}\boldsymbol{k}'_1 \,...\, \boldsymbol{k}'_l\\ \boldsymbol{k}_1 \,...\, \boldsymbol{k}_l\end{pmatrix} = \frac{1}{(N-l)!}\int\frac{\mathrm{d}^3k_{l+1}}{k_{l+1}}...\frac{\mathrm{d}^3k_N}{k_N}\rho^{s'_1...s'_ls_{l+1}...s_N}_{s_1...s_ls_{l+1}...s_N}\begin{pmatrix}\boldsymbol{k}'_1 \,...\, \boldsymbol{k}'_l\ \boldsymbol{k}_{l+1} \,...\, \boldsymbol{k}_N\\ \boldsymbol{k}_1 \,...\, \boldsymbol{k}_l\ \boldsymbol{k}_{l+1} \,...\, \boldsymbol{k}_N\end{pmatrix} \tag{27}$$

corresponds to the truncated l-particle matrix. Integrals (27) are normalized according to normalization of the matrix ρ_N, so that

$$\int\frac{\mathrm{d}^3k_1}{k_1}...\frac{\mathrm{d}^3k_l}{k_l}\rho^{s_1...s_l}_{s_1...s_l}\begin{pmatrix}\boldsymbol{k}_1 \,...\, \boldsymbol{k}_l\\ \boldsymbol{k}_1 \,...\, \boldsymbol{k}_l\end{pmatrix} = \frac{N!}{(N-l)!} = N(N-1)...(N-l+1). \tag{28}$$

Specifically, one-particle photon density matrix

$$\rho^{s'}_{s}\begin{pmatrix}\boldsymbol{k}'\\ \boldsymbol{k}\end{pmatrix} = \frac{1}{(N-1)!}\int\frac{\mathrm{d}^3k_2}{k_2}...\frac{\mathrm{d}^3k_N}{k_N}\rho^{s's_2...s_N}_{ss_2...s_N}\begin{pmatrix}\boldsymbol{k}'\ \boldsymbol{k}_2 \,...\, \boldsymbol{k}_N\\ \boldsymbol{k}\ \boldsymbol{k}_2 \,...\, \boldsymbol{k}_N\end{pmatrix} \tag{29}$$

is normalized to the number of particles

$$\int\frac{\mathrm{d}^3k}{k}\rho^{s}_{s}\begin{pmatrix}\boldsymbol{k}\\ \boldsymbol{k}\end{pmatrix} = N. \tag{30}$$

The action of the product of the annihilation and creation operators on to the density matrix is reduced to

$$\begin{aligned}\overline{a}_{(s)}(\boldsymbol{k})a_{(s')}(\boldsymbol{k}')\rho \ =\ &\int\frac{\mathrm{d}^3k'_1}{k'_1}...\frac{\mathrm{d}^3k'_N}{k'_N}\frac{\mathrm{d}^3k_1}{k_1}...\frac{\mathrm{d}^3k_N}{k_N}\sum_{j=1}^{N}k'\delta_{s's_j}\delta(\boldsymbol{k}'-\boldsymbol{k}_j)\\ &\times\ \Psi_{s_1...s_{j-1}ss_{j+1}...s_N}(\boldsymbol{k}_1,...,\boldsymbol{k}_{j-1},\boldsymbol{k},\boldsymbol{k}_{j+1},...,\boldsymbol{k}_N)\overline{\Psi}_{s'_1...s'_N}(\boldsymbol{k}'_1,...,\boldsymbol{k}'_N)\rho^{s'_1...s'_N}_{s_1...s_N}\begin{pmatrix}\boldsymbol{k}'_1 \,...\, \boldsymbol{k}'_N\\ \boldsymbol{k}_1 \,...\, \boldsymbol{k}_N\end{pmatrix}.\end{aligned} \tag{31}$$

Therefore, we get

$$\begin{aligned}&\overline{\Psi}_{s_1...s_N}(\boldsymbol{k}_1,...,\boldsymbol{k}_N)\overline{a}_{(s)}(\boldsymbol{k})a_{(s')}(\boldsymbol{k}')\ \ \rho\ \ \Psi_{s'_1...s'_N}(\boldsymbol{k}'_1,...,\boldsymbol{k}'_N)\\ &\qquad = \frac{1}{N!}\sum_{j=1}^{N}k\delta_{ss_j}\delta(\boldsymbol{k}-\boldsymbol{k}_j)\rho^{s'_1...s'_{j-1}s'_js'_{j+1}...s'_N}_{s_1...s_{j-1}s's_{j+1}...s_N}\begin{pmatrix}\boldsymbol{k}'_1 \,...\, \boldsymbol{k}'_{j-1}\ \boldsymbol{k}'_j\ \boldsymbol{k}'_{j+1} \,...\, \boldsymbol{k}'_N\\ \boldsymbol{k}_1 \,...\, \boldsymbol{k}_{j-1}\ \boldsymbol{k}'\ \boldsymbol{k}_{j+1} \,...\, \boldsymbol{k}_N\end{pmatrix}.\end{aligned} \tag{32}$$

The density matrices for particles (electrons) are introduced in a similar manner, therefore we immediately write down the expression for the joint density matrix for electrons and photons. It can be expressed through the kernel in terms of the following integral

$$\rho = \frac{1}{N!N_-!}\int \frac{\mathrm{d}^3k_1}{k_1}\frac{\mathrm{d}^3k_1'}{k_1'}...\frac{\mathrm{d}^3k_N}{k_N}\frac{\mathrm{d}^3k_N'}{k_N'}\frac{\mathrm{d}^3p_1}{p_{10}}\frac{\mathrm{d}^3p_1'}{p_{10}'}...\frac{\mathrm{d}^3p_{N_-}}{p_{N_-0}}\frac{\mathrm{d}^3p_{N_-}'}{p_{N_-0}'}\rho_{s_1...s_N,\tau_1...\tau_{N_-}}^{s_1'...s_N',\tau_1'...\tau_{N_-}'}\left(\begin{array}{c|c} \boldsymbol{k}_1' \,...\, \boldsymbol{k}_N' & \boldsymbol{p}_1' \,...\, \boldsymbol{p}_{N_-}' \\ \boldsymbol{k}_1 \,...\, \boldsymbol{k}_N & \boldsymbol{p}_1 \,...\, \boldsymbol{p}_{N_-} \end{array}\right)$$
$$\times\ \Psi_{s_1...s_N}(\boldsymbol{k}_1,...,\boldsymbol{k}_N)\overline{\Psi}_{s_1'...s_N'}(\boldsymbol{k}_1',...,\boldsymbol{k}_N')\Phi_{\tau_1...\tau_{N_-}}(\boldsymbol{p}_1,...,\boldsymbol{p}_{N_-})\Phi^{\dagger}_{\tau_1'...\tau_{N_-}'}(\boldsymbol{p}_1',...,\boldsymbol{p}_{N_-}'). \tag{33}$$

Changing the order of the photon arguments does not change anything, while changing the order of electron arguments in the state vectors or in the kernel of the density matrix changes the sign. However, if one changes simultaneously arguments in the state vectors as well as in the kernel, nothing changes since this procedure corresponds to just a change of notations. The sign does not change if one interchange lower and upper arguments, since the sign here changes even number of times. The truncated density matrix of electrons and their kernels can be introduced following the same procedure as in the case of photons. All the relations for the photon matrices are valid for the electrons too.

2.5. One-particle Distribution Functions

The kinetic equation for photons which we wish to deduce should be formulated for the one-particle polarization matrix depending on spatial coordinates, time, and photon momentum. Therefore, one has to make a transformation to such a matrix. We briefly describe the scheme for this transformation on the example of spinless particles.

Up to now we used the momentum representation where the density matrices depend on a double set of momenta and polarization indices. The spinless states could be described in the coordinate representation. Instead of the matrix with two momenta arguments, we would have a matrix with two sets of space coordinates. The transition from one representation to another can be performed by Fourier transform. For the one-particle functions such a transition is defined by the formula

$$\rho_{\mathrm{s}}(\underline{\boldsymbol{r}},\underline{\boldsymbol{r}}') = \frac{1}{(2\pi\hbar)^6}\int\frac{\mathrm{d}^3p\,\mathrm{d}^3p'}{\sqrt{p_0p_0'}}e^{-i(\underline{\boldsymbol{p}}\underline{\boldsymbol{r}}-\underline{\boldsymbol{p}}'\underline{\boldsymbol{r}}')/\hbar}\rho\begin{pmatrix}\boldsymbol{p}'\\ \boldsymbol{p}\end{pmatrix}, \tag{34}$$

where $\underline{r} = \{ct, \boldsymbol{r}\}$. The inverse transform is

$$\rho\begin{pmatrix}\boldsymbol{p}'\\ \boldsymbol{p}\end{pmatrix} = \sqrt{p_0p_0'}\int \mathrm{d}^3r\mathrm{d}^3r'e^{i(\underline{\boldsymbol{p}}\underline{\boldsymbol{r}}-\underline{\boldsymbol{p}}'\underline{\boldsymbol{r}}')/\hbar}\rho_{\mathrm{s}}(\underline{\boldsymbol{r}},\underline{\boldsymbol{r}}'). \tag{35}$$

The two factors under the root are introduced since it is convenient when transforming to our notations. In the kinetic theory, the transition from the matrix with two space arguments to the usual distribution function is done via the Wigner function (Wigner 1932; see also Silin 1971), which is defined as follows

$$\rho(\boldsymbol{p},\underline{\boldsymbol{r}}) = \int \mathrm{d}^3v e^{i\underline{\boldsymbol{p}}\underline{\boldsymbol{v}}/\hbar}\rho_{\mathrm{s}}(\underline{\boldsymbol{r}}+\underline{\boldsymbol{v}}/2,\underline{\boldsymbol{r}}-\underline{\boldsymbol{v}}/2). \tag{36}$$

The Wigner function can be also expressed through the matrix with two momentum arguments

$$\rho(\boldsymbol{p},\underline{\boldsymbol{r}}) = \frac{1}{(2\pi\hbar)^6}\int\frac{\mathrm{d}^3p_1\mathrm{d}^3p_1'}{\sqrt{p_{01}p_{01}'}}e^{-i(\underline{\boldsymbol{p}}_1-\underline{\boldsymbol{p}}_1')\underline{\boldsymbol{r}}/\hbar}\delta\left(\boldsymbol{p}-\frac{\boldsymbol{p}_1+\boldsymbol{p}_1'}{2}\right)\rho\begin{pmatrix}\boldsymbol{p}_1'\\ \boldsymbol{p}_1\end{pmatrix}. \tag{37}$$

The inverse transform reads

$$\rho\begin{pmatrix}\boldsymbol{p}_1'\\ \boldsymbol{p}_1\end{pmatrix} = \frac{\sqrt{p_{01}p_{01}'}}{(2\pi\hbar)^3}\int \mathrm{d}^3p\mathrm{d}^3re^{i(\underline{\boldsymbol{p}}_1-\underline{\boldsymbol{p}}_1')\underline{\boldsymbol{r}}/\hbar}\delta\left(\boldsymbol{p}-\frac{\boldsymbol{p}_1+\boldsymbol{p}_1'}{2}\right)\rho(\boldsymbol{p},\underline{\boldsymbol{r}}). \tag{38}$$

According to the assumption of the small scale of the interaction between photons and electrons as compared with the macroscopic scale where the distribution function in equation (38) changes significantly, one can assume that the Wigner function does not depend on the space coordinates. Then both integrals in equation (38) can be taken and the matrix becomes diagonal in momenta

$$\rho\begin{pmatrix}\boldsymbol{p}_1'\\ \boldsymbol{p}_1\end{pmatrix} = p_{01}\delta(\boldsymbol{p}_1'-\boldsymbol{p}_1)\rho(\boldsymbol{p}_1). \tag{39}$$

Here we deduced these expression in three-dimensional coordinate and momentum space, i.e. non-relativistically. In order to introduce the relativistic generalization of the Wigner function (see, e.g., de Groot, van Leeuwen, & van Weert

1980), one has to define it for the generalized momentum $\underline{p}$ which does not satisfy the relation $\underline{p}^2 = p_0^2 - \boldsymbol{p}^2 = m^2c^2$. Then one can prove that outside of the surface defined by this relation in the four-dimensional momentum space, the function strongly oscillates (Zitterbewegung) and, on average, vanishes. Further consideration leads to equation (39).

In the case of photons, situation becomes even more complicated, since the physical states with the transverse polarizations are connected to the momentum and, for the photon matrices with the fixed spin projections, transformations of the type (34) and (37) are not possible. In that case, one has to return to the general polarization states which include non-physical photons. The elimination of the later should be done after the transition to equation similar to (39). Here we also meet with the difficulties to satisfy the relation $\underline{k}^2 = 0$.

Similar difficulty appears also in the case of electrons if one takes the solutions with a given spirality (i.e., a projection of the spin on to the momentum) as the basic solutions of the Dirac equations. In that case, the transformations of the type (34) and (37) also are not possible. However, one can get solutions with the projections of the spin on to an arbitrary direction instead of the momentum. This can be achieved by the method proposed by Foldy & Wouthuysen (1950) (see also Schweber 1961). The mentioned transformations are then possible. The transition to the equation of type (39) is done in the same way as for the spinless particles and the spin indices are simply added. Without making these derivation, we take that

$$\rho_\tau^{\tau'}\begin{pmatrix}\boldsymbol{p}_1'\\ \boldsymbol{p}_1\end{pmatrix} = p_{01}\delta(\boldsymbol{p}_1' - \boldsymbol{p}_1)\rho_\tau^{\tau'}(\boldsymbol{p}_1), \tag{40}$$

where the indices characterize the spirality. For photons we make a similar approximation

$$\rho_s^{s'}\begin{pmatrix}\boldsymbol{k}'\\ \boldsymbol{k}\end{pmatrix} = k\delta(\boldsymbol{k}' - \boldsymbol{k})\rho_s^{s'}(\boldsymbol{k}), \tag{41}$$

The matrices $\rho_\tau^{\tau'}(\boldsymbol{p})$ and $\rho_s^{s'}(\boldsymbol{k})$ depend, of course, on time and space coordinates, but this macroscopic dependence is unimportant for the description of a single scattering act.

In the normalization condition (30) the momenta $\boldsymbol{k}$ and $\boldsymbol{k}'$ are equal to each other. Formally, for equal momenta the δ-function in equation (41) becomes infinite. However, since we operate with the quantities in the finite volume (basic box), according to (2) one has to take

$$\delta(\boldsymbol{k} - \boldsymbol{k}) = \frac{V}{(2\pi\hbar)^3}, \quad \rho_s^{s'}\begin{pmatrix}\boldsymbol{k}\\ \boldsymbol{k}\end{pmatrix} = k\frac{V}{(2\pi\hbar)^3}\rho_s^{s'}(\boldsymbol{k}). \tag{42}$$

Then the normalization condition (30) reads

$$\int \frac{\mathrm{d}^3k}{k}\rho_s^s\begin{pmatrix}\boldsymbol{k}\\ \boldsymbol{k}\end{pmatrix} = N = \frac{V}{(2\pi\hbar)^3}\int \mathrm{d}^3k\,\rho_s^s(\boldsymbol{k}), \quad \text{so that} \quad \frac{1}{(2\pi\hbar)^3}\int \mathrm{d}^3k\,\rho_s^s(\boldsymbol{k}) = \frac{N}{V}. \tag{43}$$

When simultaneously $V \to \infty$ and $N \to \infty$ while their ratio is constant, relation (43) transforms to the usual condition normalizing the polarization matrix. The sum $\rho_s^s(\boldsymbol{k}) = \rho_1^1(\boldsymbol{k}) + \rho_2^2(\boldsymbol{k})$ is equal to the double mean occupation number of the photon states $2\rho(\boldsymbol{k})$. This agrees with the fact that unpolarized radiation is described by the matrix

$$\rho_s^{s'}(\boldsymbol{k}) = \delta_{ss'}\rho(\boldsymbol{k}), \tag{44}$$

so that $\rho_s^s(\boldsymbol{k}) = 2\rho(\boldsymbol{k})$.

The normalization condition for the electron polarization matrix can be obtained from the equations analogous to (30) and (2). The later takes the form $\delta(\boldsymbol{p} - \boldsymbol{p}) = V/(2\pi\hbar)^3$, so that

$$\frac{1}{(2\pi\hbar)^3}\int \mathrm{d}^3p\,\rho_\tau^\tau(\boldsymbol{p}) = \frac{N_-}{V}. \tag{45}$$

The unpolarized electrons are characterized by the diagonal matrix

$$\rho_\tau^{\tau'}(\boldsymbol{p}) = \frac{(2\pi\hbar)^3}{2}\delta_\tau^{\tau'} f_{\mathrm{e}}(\boldsymbol{p}), \tag{46}$$

where the distribution function in the comoving frame is normalized to the electron number density

$$\int \mathrm{d}^3p f_{\mathrm{e}}(\boldsymbol{p}) = \frac{N_-}{V}. \tag{47}$$

We use the same notation, ρ, for both photon and electron matrices, but they differ by their arguments and indices.

In spite of the fact that equations (40) and (41) are deduced in a non-relativistic way they are relativistically covariant. Normalization conditions (43) and (45) are written in the reference frames where the photon and electron gases are at rest. The corresponding relativistically covariant equations in an arbitrary frames are

$$\frac{1}{(2\pi\hbar)^3}\int\frac{\mathrm{d}^3k}{k}\underline{k}\rho^s_s(\boldsymbol{k})=\frac{N}{V}\langle\underline{k}\rangle,\quad \frac{1}{(2\pi\hbar)^3}\int\frac{\mathrm{d}^3p}{p_0}\underline{p}\rho^\tau_\tau(\boldsymbol{p})=\frac{N_-}{V}\langle\underline{p}\rangle, \tag{48}$$

where $\langle\underline{k}\rangle$ and $\langle\underline{p}\rangle$ are the average photon and electron momenta in a given frame.

Let us note that the interaction time T_0 is also related to the δ-function. This relation is the relativistic counterpart of (2), i.e. its time-like form

$$\delta(k-k)=\frac{cT_0}{2\pi\hbar}. \tag{49}$$

3. Interaction between Photons and Electrons

3.1. Equation for the Density Matrix

Density matrix (33) satisfies the quantum Liouville equation (Landau & Lifshitz 1977; Silin 1971; de Groot et al. 1980)

$$i\hbar\frac{\partial\rho(t)}{\partial t}=H(t)\rho(t)-\rho(t)H(t) \tag{50}$$

where $H(t)$ is a Hamiltonian. This equation can be transformed to explicitly covariant form if we proceed from the Tomonaga–Schwinger equation (Bogoliubov & Shirkov 1959). In perturbation theory, the solution of this equation can be written as a series. It is shown in the text books on quantum electrodynamics that a part of this series corresponding to the scattering of a photon by an electron can be represented in the following form (index CE stands for Compton scattering by an Electron)

$$U^{\mathrm{CE}}_{\mathrm{V}}(t,t_0)=\int\limits_{t_0}^{t}\mathrm{d}t'\int\limits_{t_0}^{t}\mathrm{d}t''\int\limits_{V}\mathrm{d}^3r'\int\limits_{V}\mathrm{d}^3r''\mathcal{S}^{\mathrm{CE}}(\underline{r}',\underline{r}'')=\frac{1}{c^2}\int\limits_{\mathcal{V}}\mathrm{d}^4r'\int\limits_{\mathcal{V}}\mathrm{d}^4r''\mathcal{S}^{\mathrm{CE}}(\underline{r}',\underline{r}''), \tag{51}$$

where $\mathcal{V}$ is part of Minkowsky space $ct_0\le x_0\le ct$, $-L_{\mathrm{x}}/2\le x\le L_{\mathrm{z}}/2$, $-L_{\mathrm{y}}/2\le y\le L_{\mathrm{y}}/2$,$-L_{\mathrm{z}}/2\le z\le L_{\mathrm{z}}/2$, and

$$\mathcal{S}^{\mathrm{CE}}(\underline{r}',\underline{r}'')=i\frac{e^2c}{(2\pi\hbar)^9}\int\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p'}{p'_0}\frac{\mathrm{d}^3k}{k}\frac{\mathrm{d}^3k'}{k'}b^\dagger_{\tau'}(\boldsymbol{p}')b_\tau(\boldsymbol{p})\overline{a}_{(s')}(\boldsymbol{k}')a_{(s)}(\boldsymbol{k})\mathcal{N}^{s'\tau'}_{s\;\tau}\left(\begin{array}{c|c|c}\boldsymbol{k}'&\boldsymbol{p}'&\underline{r}'\\ \boldsymbol{k}&\boldsymbol{p}&\underline{r}''\end{array}\right). \tag{52}$$

In the last expression

$$\mathcal{N}^{s'\tau'}_{s\;\tau}\left(\begin{array}{c|c|c}\boldsymbol{k}'&\boldsymbol{p}'&\underline{r}'\\ \boldsymbol{k}&\boldsymbol{p}&\underline{r}''\end{array}\right)=\frac{1}{2}\left[\mathcal{M}^{s'\tau'}_{s\;\tau}\left(\begin{array}{c|c|c}\boldsymbol{k}'&\boldsymbol{p}'&\underline{r}'\\ \boldsymbol{k}&\boldsymbol{p}&\underline{r}''\end{array}\right)+\mathcal{M}^{s'\tau'}_{s\;\tau}\left(\begin{array}{c|c|c}\boldsymbol{k}'&\boldsymbol{p}'&\underline{r}''\\ \boldsymbol{k}&\boldsymbol{p}&\underline{r}'\end{array}\right)\right] \tag{53}$$

and

$$\begin{aligned}\mathcal{M}^{s'\tau'}_{s\;\tau}\left(\begin{array}{c|c|c}\boldsymbol{k}'&\boldsymbol{p}'&\underline{r}'\\ \boldsymbol{k}&\boldsymbol{p}&\underline{r}''\end{array}\right)&=mc\int\mathrm{d}^4p_{\mathrm{v}}\overline{u}^{\tau'}(\boldsymbol{p}')\left[\hat{e}_{(s')}(\boldsymbol{k}')\frac{mc+\hat{p}_{\mathrm{v}}}{m^2c^2-\underline{p}^2_{\mathrm{v}}-i0}\hat{e}_{(s)}(\boldsymbol{k})e^{i(\underline{k}'\underline{r}'-\underline{k}\,\underline{r}'')/\hbar}\right.\\ &+\left.\hat{e}_{(s)}(\boldsymbol{k})\frac{mc+\hat{p}_{\mathrm{v}}}{m^2c^2-\underline{p}^2_{\mathrm{v}}-i0}\hat{e}_{(s')}(\boldsymbol{k}')e^{i(\underline{k}'\underline{r}''-\underline{k}\,\underline{r}')/\hbar}\right]e^{i(\underline{p}'\underline{r}'-\underline{p}\underline{r}'')/\hbar}e^{-i\underline{p}_{\mathrm{v}}(\underline{r}'-\underline{r}'')/\hbar}u^\tau(\boldsymbol{p}).\end{aligned} \tag{54}$$

Here $u^\tau(\boldsymbol{p})$, the elementary solutions of the Dirac equation describing electron, are the columns containing four functions, $\overline{u}^\tau(\boldsymbol{p})=[u^\tau(\boldsymbol{p})]^\dagger\gamma^0$ and $\hat{p}=\underline{p}\,\underline{\gamma}=p_0\gamma^0-\boldsymbol{p}\cdot\boldsymbol{\gamma}=p_\mu\gamma^\mu$, where γ^μ are the Dirac matrices (e.g. Schweber 1961; Berestetskii, Lifshitz, & Pitaevskii 1982). The scalar product $\underline{k}\,\underline{r}=kct-\boldsymbol{k}\cdot\boldsymbol{r}$. The imaginary term $i0$ determines the definite rule of circuit of singularity in the denominator. It is obvious that matrices (52) and (53) are symmetric relative to $\underline{r}'$ and $\underline{r}''$.

If $V\to\infty$, $t_0\to-\infty$, $t\to\infty$, the electron Compton operator $U^{\mathrm{CE}}_\infty(-\infty,\infty)=S^{\mathrm{CE}}$ contains two integrals over four-dimensional space ($\underline{r}'$ and $\underline{r}''$). Two four-dimensional δ-functions appear after calculating these integrals. One of them disappears when we take the integral over four-dimensional momentum of virtual electron $\underline{p}_{\mathrm{v}}$, whereas the second one reflects the conservation laws. Thus integrals over the whole time-space ($x_0=ct$)

$$\int\mathrm{d}^4r'\int\mathrm{d}^4r''\mathcal{N}^{s'\tau'}_{s\;\tau}\left(\begin{array}{c|c|c}\boldsymbol{k}'&\boldsymbol{p}'&\underline{r}'\\ \boldsymbol{k}&\boldsymbol{p}&\underline{r}''\end{array}\right)=\int\mathrm{d}^4r'\int\mathrm{d}^4r''\mathcal{M}^{s'\tau'}_{s\;\tau}\left(\begin{array}{c|c|c}\boldsymbol{k}'&\boldsymbol{p}'&\underline{r}'\\ \boldsymbol{k}&\boldsymbol{p}&\underline{r}''\end{array}\right)=(2\pi\hbar)^8\delta(\underline{p}'+\underline{k}'-\underline{p}-\underline{k})M^{s'\tau'}_{s\;\tau}\left(\begin{array}{c|c}\boldsymbol{k}'&\boldsymbol{p}'\\ \boldsymbol{k}&\boldsymbol{p}\end{array}\right), \tag{55}$$

where the amplitude of the process

$$M^{s'\tau'}_{s\ \tau}\left(\begin{matrix}\boldsymbol{k}'\\ \boldsymbol{k}\end{matrix}\middle|\begin{matrix}\boldsymbol{p}'\\ \boldsymbol{p}\end{matrix}\right) = mc\overline{u}^{\tau'}(\boldsymbol{p}')\left[\hat{e}_{(s')}(\boldsymbol{k}')\frac{mc+\hat{p}+\hat{k}}{m^2c^2-(\underline{p}+\underline{k})^2}\hat{e}_{(s)}(\boldsymbol{k}) + \hat{e}_{(s)}(\boldsymbol{k})\frac{mc+\hat{p}-\hat{k}'}{m^2c^2-(\underline{p}-\underline{k}')^2}\hat{e}_{(s')}(\boldsymbol{k}')\right]u^{\tau}(\boldsymbol{p}). \tag{56}$$

The amplitude has an obvious property

$$\left[M^{s'\tau'}_{s\ \tau}\left(\begin{matrix}\boldsymbol{k}'\\ \boldsymbol{k}\end{matrix}\middle|\begin{matrix}\boldsymbol{p}'\\ \boldsymbol{p}\end{matrix}\right)\right]^* = M^{s\ \tau}_{s'\tau'}\left(\begin{matrix}\boldsymbol{k}\\ \boldsymbol{k}'\end{matrix}\middle|\begin{matrix}\boldsymbol{p}\\ \boldsymbol{p}'\end{matrix}\right). \tag{57}$$

Then for matrix $S^{\rm CE}$ the following expression is obtained (Bogoliubov & Shirkov 1959)

$$S^{\rm CE} = \frac{e^2}{\hbar c}\frac{i}{2\pi}\int\frac{{\rm d}^3k}{k}\frac{{\rm d}^3k'}{k'}\frac{{\rm d}^3p}{p_0}\frac{{\rm d}^3p'}{p_0'}\delta(\underline{p}'+\underline{k}'-\underline{p}-\underline{k})b^\dagger_{\tau'}(\boldsymbol{p}')b_\tau(\boldsymbol{p})\overline{a}_{(s')}(\boldsymbol{k}')a_{(s)}(\boldsymbol{k})M^{s'\tau'}_{s\ \tau}\left(\begin{matrix}\boldsymbol{k}'\\ \boldsymbol{k}\end{matrix}\middle|\begin{matrix}\boldsymbol{p}'\\ \boldsymbol{p}\end{matrix}\right). \tag{58}$$

It is easy to show that the operators $U^{\rm CE}_{\rm V}(t,t_0)$ and $S^{\rm CE}$ being conjugated only change the sign.

If we omit all terms which are higher than the second order and those which do not conserve the number and/or the quality of particles then the equation for the photon-electron density matrix describing Compton scattering by electrons takes the form

$$\int\limits_{t_0}^{t}\frac{\partial\rho(t')}{\partial t'}{\rm d}t' = \rho(t)-\rho(t_0) = U^{\rm CE}_{\rm V}(t,t_0)\rho(t_0)+\rho(t_0)U^{\rm CE\,\dagger}_{\rm V}(t,t_0). \tag{59}$$

The derivative in equation (59) describes all variations of the density matrix which have very different scales (hierarchy of scales). The duration and the characteristic scale of the interaction are much less than the scales of macroscopic changes of the matrix and this fact is the basis of the Bogoliubov method of deducing the kinetic equations. Thus we assume that the density matrix ρ changes negligibly during the interaction. Following Baranger (1958), we take the value of the derivative at time t out of the integral in the lhs of equation (59), i.e. the lhs is now $T_0{\rm d}\rho/{\rm d}t$. We write the full derivative because it describes now macroscopic changes. Thus we get the initial basic equation

$$T_0\frac{{\rm d}\rho(t)}{{\rm d}t} = \rho(t)-\rho(t_0) = U^{\rm CE}_{\rm V}(t,t_0)\rho(t_0)-\rho(t_0)U^{\rm CE}_{\rm V}(t,t_0). \tag{60}$$

3.2. Equation for the Kernel of the Density Matrix

Let us write equation (60) not in the operator form, but via the kernel of the photon-electron matrix keeping temporarily the number of components fixed (specifically, N photons and N_- electrons). Simultaneously we substitute the time argument t of the density matrix instead of t_0, because this argument is macroscopic time.

According to equation (26) for photons and analogous for electrons, we multiply equation (60) by the corresponding state vectors and their conjugates on the left and right. We get

$$\begin{aligned}
&T_0\ \frac{\rm d}{{\rm d}t}\rho^{s_1'\ldots s_N',\tau_1'\ldots\tau_{N_-}'}_{s_1\ldots s_N,\tau_1\ldots\tau_{N_-}}\left(\begin{matrix}\boldsymbol{k}_1'\ \ldots\ \boldsymbol{k}_N'\\ \boldsymbol{k}_1\ \ldots\ \boldsymbol{k}_N\end{matrix}\middle|\begin{matrix}\boldsymbol{p}_1'\ \ldots\ \boldsymbol{p}_{N_-}'\\ \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}_{N_-}\end{matrix}\middle|t\right) = \overline{\Psi}_{s_1\ldots s_N}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)\Phi^\dagger_{\tau_1\ldots\tau_{N_-}}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_{N_-})\\
&\times\ \left[U^{\rm CE}_{\rm V}(t,t_0)\rho(t)-\rho(t)U^{\rm CE}_{\rm V}(t,t_0)\right]\Psi_{s_1'\ldots s_N'}(\boldsymbol{k}_1',\ldots,\boldsymbol{k}_N')\Phi_{\tau_1'\ldots\tau_{N_-}'}(\boldsymbol{p}_1',\ldots,\boldsymbol{p}_{N_-}')N!N_-!\\
&=\ i\frac{e^2}{c(2\pi\hbar)^9}\overline{\Psi}_{s_1\ldots s_N}(\boldsymbol{k}_1\ldots\boldsymbol{k}_N)\Phi^\dagger_{\tau_1\ldots\tau_{N_-}}(\boldsymbol{p}_1\ldots\boldsymbol{p}_{N_-})\int\limits_{\mathcal{V}}{\rm d}^4r'\int\limits_{\mathcal{V}}{\rm d}^4r''\int\frac{{\rm d}^3k}{k}\frac{{\rm d}^3k'}{k'}\frac{{\rm d}^3p}{p_0}\frac{{\rm d}^3p'}{p_0'}\mathcal{N}^{s'\tau'}_{s\ \tau}\left(\begin{matrix}\boldsymbol{k}'\\ \boldsymbol{k}\end{matrix}\middle|\begin{matrix}\boldsymbol{p}'\\ \boldsymbol{p}\end{matrix}\middle|\begin{matrix}\underline{r}'\\ \underline{r}''\end{matrix}\right)\\
&\times\ \int\frac{{\rm d}^3k_1''}{k_1''}\frac{{\rm d}^3k_1'''}{k_1'''}\ldots\frac{{\rm d}^3k_N''}{k_N''}\frac{{\rm d}^3k_N'''}{k_N'''}\frac{{\rm d}^3p_1''}{p_{01}''}\frac{{\rm d}^3p_1'''}{p_{01}'''}\ldots\frac{{\rm d}^3p_{N_-}''}{p_{0N_-}''}\frac{{\rm d}^3p_{N_-}'''}{p_{0N_-}'''}\rho^{s_1'''\ldots s_N''',\tau_1'''\ldots\tau_{N_-}'''}_{s_1''\ldots s_N'',\tau_1''\ldots\tau_{N_-}''}\left(\begin{matrix}\boldsymbol{k}_1'''\ \ldots\ \boldsymbol{k}_N'''\\ \boldsymbol{k}_1''\ \ldots\ \boldsymbol{k}_N''\end{matrix}\middle|\begin{matrix}\boldsymbol{p}_1'''\ \ldots\ \boldsymbol{p}_{N_-}'''\\ \boldsymbol{p}_1''\ \ldots\ \boldsymbol{p}_{N_-}''\end{matrix}\middle|t\right)\\
&\times\ \Big[b^\dagger_{\tau'}(\boldsymbol{p}')b_\tau(\boldsymbol{p})\overline{a}_{(s')}(\boldsymbol{k}')a_{(s)}(\boldsymbol{k})\Psi_{s_1''\ldots s_N''}(\boldsymbol{k}_1'',\ldots,\boldsymbol{k}_N'')\Phi_{\tau_1''\ldots\tau_{N_-}''}(\boldsymbol{p}_1'',\ldots,\boldsymbol{p}_{N_-}'')\overline{\Psi}_{s_1'''\ldots s_N'''}(\boldsymbol{k}_1''',\ldots,\boldsymbol{k}_N''')\Phi^\dagger_{\tau_1'''\ldots\tau_{N_-}'''}(\boldsymbol{p}_1''',\ldots,\boldsymbol{p}_{N_-}''')\\
&-\ \Psi_{s_1''\ldots s_N''}(\boldsymbol{k}_1'',\ldots,\boldsymbol{k}_N'')\Phi_{\tau_1''\ldots\tau_{N_-}''}(\boldsymbol{p}_1'',\ldots,\boldsymbol{p}_{N_-}'')\overline{\Psi}_{s_1'''\ldots s_N'''}(\boldsymbol{k}_1''',\ldots,\boldsymbol{k}_N''')\Phi^\dagger_{\tau_1'''\ldots\tau_{N_-}'''}(\boldsymbol{p}_1''',\ldots,\boldsymbol{p}_{N_-}''')\\
&\times\ b^\dagger_\tau(\boldsymbol{p})b_{\tau'}(\boldsymbol{p}')\overline{a}_{(s)}(\boldsymbol{k})a_{(s')}(\boldsymbol{k}')\Big]\Psi_{s_1'\ldots s_N'}(\boldsymbol{k}_1',\ldots,\boldsymbol{k}_N')\Phi_{\tau_1'\ldots\tau_{N_-}'}(\boldsymbol{p}_1',\ldots,\boldsymbol{p}_{N_-}').
\end{aligned} \tag{61}$$

Using the normalization conditions (10) and (21) for the state vectors as well as formulae (13) and (20) describing the action of the creation and annihilation operators on to the state vectors, and integrating over all photon and electron momenta with two and three primes, we get

$$T_0\,\frac{\mathrm{d}}{\mathrm{d}t}\rho^{s'_1\ldots s'_N,\tau'_1\ldots\tau'_{N_-}}_{s_1\ldots s_N,\tau_1\ldots\tau_{N_-}}\left(\begin{array}{c|c}\boldsymbol{k}'_1\ \ldots\ \boldsymbol{k}'_N & \boldsymbol{p}'_1\ \ldots\ \boldsymbol{p}'_{N_-}\\ \boldsymbol{k}_1\ \ldots\ \boldsymbol{k}_N & \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}_{N_-}\end{array}\middle|\,t\right) = i\frac{e^2}{c(2\pi\hbar)^9}\int\limits_{\mathcal{V}}\mathrm{d}^4r'\int\limits_{\mathcal{V}}\mathrm{d}^4r''\int\frac{\mathrm{d}^3k}{k}\frac{\mathrm{d}^3k'}{k'}\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p'}{p'_0}\mathcal{N}^{s'\,\tau'}_{s\,\tau}\left(\begin{array}{c|c|c}\boldsymbol{k}' & \boldsymbol{p}' & \underline{r}'\\ \boldsymbol{k} & \boldsymbol{p} & \underline{r}''\end{array}\right)$$

$$\times\sum_{r=1}^{N}\sum_{j=1}^{N_-}\Bigg[k'\delta_{s's_r}\delta(\boldsymbol{k}'-\boldsymbol{k}_r)p'_0\delta_{\tau'\tau_j}\delta(\boldsymbol{p}'-\boldsymbol{p}_j)\rho^{s'_1\ldots s'_{r-1}s'_r s'_{r+1}\ldots s'_N,\tau'_1\ldots\tau'_{j-1}\tau'_j\tau'_{j+1}\ldots\tau'_{N_-}}_{s_1\ldots s_{r-1}s\ s_{r+1}\ldots s_N,\tau_1\ldots\tau_{j-1}\tau\ \tau_{j+1}\ldots\tau_{N_-}}\left(\begin{array}{c|c}\boldsymbol{k}'_1\ \ldots\ \boldsymbol{k}'_r\ \ldots\ \boldsymbol{k}'_N & \boldsymbol{p}'_1\ \ldots\ \boldsymbol{p}'_j\ \ldots\ \boldsymbol{p}'_{N_-}\\ \boldsymbol{k}_1\ \ldots\ \boldsymbol{k}\ \ldots\ \boldsymbol{k}_N & \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}\ \ldots\ \boldsymbol{p}_{N_-}\end{array}\middle|\,t\right)$$

$$-\ k\delta_{ss'_r}\delta(\boldsymbol{k}-\boldsymbol{k}'_r)p_0\delta_{\tau\tau'_j}\delta(\boldsymbol{p}-\boldsymbol{p}'_j)\rho^{s'_1\ldots s'_{r-1}s'\ s'_{r+1}\ldots s'_N,\tau'_1\ldots\tau'_{j-1}\tau'\tau'_{j+1}\ldots\tau'_{N_-}}_{s_1\ldots s_{r-1}s_r s_{r+1}\ldots s_N,\tau_1\ldots\tau_{j-1}\tau_j\tau_{j+1}\ldots\tau_{N_-}}\left(\begin{array}{c|c}\boldsymbol{k}'_1\ \ldots\ \boldsymbol{k}'\ \ldots\ \boldsymbol{k}'_N & \boldsymbol{p}'_1\ \ldots\ \boldsymbol{p}'\ \ldots\ \boldsymbol{p}'_{N_-}\\ \boldsymbol{k}_1\ \ldots\ \boldsymbol{k}_r\ \ldots\ \boldsymbol{k}_N & \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}_j\ \ldots\ \boldsymbol{p}_{N_-}\end{array}\middle|\,t\right)\Bigg]. \tag{62}$$

Deriving this equation, we used a number of times the possibility to change the summation order between rows and columns (see derivation of eq. [10]) and as a result the factorials canceled out. The difference of two values of density matrix in equation (60) gives the difference of corresponding matrix elements which we do not write.

3.3. Equations for the One-photon Matrix and the Correlation Matrix

Let us write now the equation for the kernel of the truncated photon matrix of the first order which follows from equation (62). We equate all corresponding upper and lower polarization indices and momenta of electrons and all (except the first) indices and momenta of photons, i.e. we sum over indices and integrate over momenta. The result, according to the definition (29), is then divided by $(N-1)!N_-!$

$$T_0\,\frac{\mathrm{d}}{\mathrm{d}t}\rho^{s'_1}_{s_1}\left(\begin{array}{c}\boldsymbol{k}'_1\\ \boldsymbol{k}_1\end{array}\middle|\,t\right) = i\frac{e^2}{c(2\pi\hbar)^9}\int\limits_{\mathcal{V}}\mathrm{d}^4r'\int\limits_{\mathcal{V}}\mathrm{d}^4r''\int\frac{\mathrm{d}^3k}{k}\frac{\mathrm{d}^3k'}{k'}\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p'}{p'_0}\mathcal{N}^{s'\,\tau'}_{s\,\tau}\left(\begin{array}{c|c|c}\boldsymbol{k}' & \boldsymbol{p}' & \underline{r}'\\ \boldsymbol{k} & \boldsymbol{p} & \underline{r}''\end{array}\right)\int\frac{\mathrm{d}^3k_2}{k_2}\ldots\frac{\mathrm{d}^3k_N}{k_N}\frac{\mathrm{d}^3p_1}{p_{01}}\ldots\frac{\mathrm{d}^3p_{N_-}}{p_{0N_-}}$$

$$\times\ \frac{1}{(N-1)!N_-!}\Bigg[k'\delta_{s's_1}\delta(\boldsymbol{k}'-\boldsymbol{k}_1)\sum_{j=1}^{N_-}p'_0\delta_{\tau'\tau_j}\delta(\boldsymbol{p}'-\boldsymbol{p}_j)\rho^{s'_1 s_2\ldots s_N,\tau_1\ldots\tau_j\ldots\tau_{N_-}}_{s\ \ s_2\ldots s_N,\tau_1\ldots\tau\ \ldots\tau_{N_-}}\left(\begin{array}{c|c}\boldsymbol{k}'_1\boldsymbol{k}_2\ \ldots\ \boldsymbol{k}_N & \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}_j\ \ldots\ \boldsymbol{p}_{N_-}\\ \boldsymbol{k}\ \ \boldsymbol{k}_2\ \ldots\ \boldsymbol{k}_N & \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}\ \ldots\ \boldsymbol{p}_{N_-}\end{array}\middle|\,t\right)$$

$$+\ \sum_{r=2}^{N}k'\delta_{s's_r}\delta(\boldsymbol{k}'-\boldsymbol{k}_r)\sum_{j=1}^{N_-}p'_0\delta_{\tau'\tau_j}\delta(\boldsymbol{p}'-\boldsymbol{p}_j)\rho^{s'_1 s_2\ldots s_r\ldots s_N,\tau_1\ldots\tau_j\ldots\tau_{N_-}}_{s_1 s_2\ldots s\ \ \ldots s_N,\tau_1\ldots\tau\ \ldots\tau_{N_-}}\left(\begin{array}{c|c}\boldsymbol{k}'_1\boldsymbol{k}_2\ \ldots\ \boldsymbol{k}_r\ \ldots\ \boldsymbol{k}_N & \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}_j\ \ldots\ \boldsymbol{p}_{N_-}\\ \boldsymbol{k}\ \ \boldsymbol{k}_2\ \ldots\ \boldsymbol{k}\ \ldots\ \boldsymbol{k}_N & \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}\ \ldots\ \boldsymbol{p}_{N_-}\end{array}\middle|\,t\right)$$

$$-\ k\delta_{ss'_1}\delta(\boldsymbol{k}-\boldsymbol{k}'_1)\sum_{j=1}^{N_-}p_0\delta_{\tau\tau_j}\delta(\boldsymbol{p}-\boldsymbol{p}'_j)\rho^{s'\ s_2\ldots s_N,\tau_1\ldots\tau'\ldots\tau_{N_-}}_{s_1 s_2\ldots s_N,\tau_1\ldots\tau_j\ldots\tau_{N_-}}\left(\begin{array}{c|c}\boldsymbol{k}'\boldsymbol{k}_2\ \ldots\ \boldsymbol{k}_N & \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}'\ \ldots\ \boldsymbol{p}_{N_-}\\ \boldsymbol{k}_1\boldsymbol{k}_2\ \ldots\ \boldsymbol{k}_N & \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}_j\ \ldots\ \boldsymbol{p}_{N_-}\end{array}\middle|\,t\right)$$

$$-\ \sum_{r=2}^{N}k\delta_{ss_r}\delta(\boldsymbol{k}-\boldsymbol{k}_r)\sum_{j=1}^{N_-}p_0\delta_{\tau\tau_j}\delta(\boldsymbol{p}-\boldsymbol{p}_j)\rho^{s'_1 s_2\ldots s'\ \ldots s_N,\tau_1\ldots\tau'\ldots\tau_{N_-}}_{s_1 s_2\ldots s_r\ldots s_N,\tau_1\ldots\tau_j\ldots\tau_{N_-}}\left(\begin{array}{c|c}\boldsymbol{k}'_1\boldsymbol{k}_2\ \ldots\ \boldsymbol{k}'\ \ldots\ \boldsymbol{k}_N & \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}'\ \ldots\ \boldsymbol{p}_{N_-}\\ \boldsymbol{k}_1\boldsymbol{k}_2\ \ldots\ \boldsymbol{k}_r\ \ldots\ \boldsymbol{k}_N & \boldsymbol{p}_1\ \ldots\ \boldsymbol{p}_j\ \ldots\ \boldsymbol{p}_{N_-}\end{array}\middle|\,t\right)\Bigg]. \tag{63}$$

Here in summation over r we separately write the first term corresponding to $r=1$. Since every integral in the sums over r and over j gives the same result, the summation is reduced to the multiplication by the number of terms in the sums, so that the factorials cancel out. The sums over $r=2,\ldots,N$ give the same kernel of the truncated matrix $\rho^{s'_1 s',\tau'}_{s_1 s,\ \tau}$ and cancel out. The resulting equation in the right hand side contains only kernel of the matrix depending on variables of one photon and one electron

$$T_0\frac{\mathrm{d}}{\mathrm{d}t}\rho^{s'_1}_{s_1}\left(\begin{array}{c}\boldsymbol{k}'_1\\ \boldsymbol{k}_1\end{array}\middle|\,t\right) = i\frac{e^2}{c(2\pi\hbar)^9}\int\limits_{\mathcal{V}}\mathrm{d}^4r'\int\limits_{\mathcal{V}}\mathrm{d}^4r''\int\frac{\mathrm{d}^3k}{k}\frac{\mathrm{d}^3k'}{k'}\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p'}{p'_0}\mathcal{N}^{s'\,\tau'}_{s\,\tau}\left(\begin{array}{c|c|c}\boldsymbol{k}' & \boldsymbol{p}' & \underline{r}'\\ \boldsymbol{k} & \boldsymbol{p} & \underline{r}''\end{array}\right)$$

$$\times\ \left[k'\delta_{s's_1}\delta(\boldsymbol{k}'-\boldsymbol{k}_1)\rho^{s'_1,\tau'}_{s,\ \tau}\left(\begin{array}{c|c}\boldsymbol{k}'_1 & \boldsymbol{p}'\\ \boldsymbol{k} & \boldsymbol{p}\end{array}\middle|\,t\right) - k\delta_{ss'_1}\delta(\boldsymbol{k}-\boldsymbol{k}'_1)\rho^{s',\tau'}_{s_1,\tau}\left(\begin{array}{c|c}\boldsymbol{k}' & \boldsymbol{p}'\\ \boldsymbol{k}_1 & \boldsymbol{p}\end{array}\middle|\,t\right)\right]. \tag{64}$$

Let us now derive the equation for $\rho^{s'_1,\tau'_1}_{s_1,\tau_1}$. We follow the same procedure as when deriving equation (64), fixing the characteristics (momenta and indices) of one photon and one electron in equation (62). One has to separate these variables when we consider the action of annihilation and creation operators on them. The resulting expression contains four couples of terms with pluses and minuses. The first couple does not have summation at all (i.e., corresponds to $r=1, j=1$), the second and the third couples contain summation over the photon and electron variables, respectively, and the fourth couple contains summations over variables of both interacting particles starting from $r=2$ and $j=2$. The terms in the last couple (with double sums) give the same function $\rho^{s'_1 s',\tau'_1\tau'}_{s_1 s,\tau_1\tau}$ and annihilate. We take as the lhs of the equation the difference which stands on the second place of equation (60). We omit here the argument t of the density matrices in the rhs of the equation based on the assumption formulated in the next section. We get

$$\begin{aligned}
\rho^{s_1',\tau_1'}_{s_1,\tau_1}\left(\begin{array}{c|c|c} \boldsymbol{k}_1' & \boldsymbol{p}_1' & \\ \boldsymbol{k}_1 & \boldsymbol{p}_1 & \end{array} t\right) &- \rho^{s_1',\tau_1'}_{s_1,\tau_1}\left(\begin{array}{c|c|c} \boldsymbol{k}_1' & \boldsymbol{p}_1' & \\ \boldsymbol{k}_1 & \boldsymbol{p}_1 & \end{array} t_0\right) = i\frac{e^2}{c(2\pi\hbar)^9}\int\limits_{\mathcal{V}}\mathrm{d}^4r'\int\limits_{\mathcal{V}}\mathrm{d}^4r''\int\frac{\mathrm{d}^3k}{k}\frac{\mathrm{d}^3k'}{k'}\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p'}{p_0'}\mathcal{N}^{s'\,\tau'}_{s\;\tau}\left(\begin{array}{c|c|c} \boldsymbol{k}' & \boldsymbol{p}' & \underline{r}' \\ \boldsymbol{k} & \boldsymbol{p} & \underline{r}'' \end{array}\right) \\
&\times\ \Bigg[k'\delta_{s's_1}\delta(\boldsymbol{k}'-\boldsymbol{k}_1)p_0'\delta_{\tau'\tau_1}\delta(\boldsymbol{p}'-\boldsymbol{p}_1)\rho^{s_1',\tau_1'}_{s,\tau}\left(\begin{array}{c|c} \boldsymbol{k}_1' & \boldsymbol{p}_1' \\ \boldsymbol{k} & \boldsymbol{p} \end{array}\right) - k\delta_{ss_1'}\delta(\boldsymbol{k}-\boldsymbol{k}_1')p_0\delta_{\tau\tau_1'}\delta(\boldsymbol{p}-\boldsymbol{p}_1')\rho^{s',\tau'}_{s_1,\tau_1}\left(\begin{array}{c|c} \boldsymbol{k}' & \boldsymbol{p}' \\ \boldsymbol{k}_1 & \boldsymbol{p}_1 \end{array}\right) \\
&+\ k'\delta_{s's_1}\delta(\boldsymbol{k}'-\boldsymbol{k}_1)\rho^{s_1',\tau_1'\tau'}_{s,\tau_1\tau}\left(\begin{array}{c|c} \boldsymbol{k}_1' & \boldsymbol{p}_1'\ \boldsymbol{p}' \\ \boldsymbol{k} & \boldsymbol{p}_1\ \boldsymbol{p} \end{array}\right) - k\delta_{ss_1'}\delta(\boldsymbol{k}-\boldsymbol{k}_1')\rho^{s',\tau_1'\tau'}_{s_1,\tau_1\tau}\left(\begin{array}{c|c} \boldsymbol{k}' & \boldsymbol{p}_1'\ \boldsymbol{p}' \\ \boldsymbol{k}_1 & \boldsymbol{p}_1\ \boldsymbol{p} \end{array}\right) \\
&+\ p_0'\delta_{\tau'\tau_1}\delta(\boldsymbol{p}'-\boldsymbol{p}_1)\rho^{s_1's',\tau_1'}_{s_1s,\tau}\left(\begin{array}{c|c} \boldsymbol{k}_1'\ \boldsymbol{k}' & \boldsymbol{p}_1' \\ \boldsymbol{k}_1\ \boldsymbol{k} & \boldsymbol{p} \end{array}\right) - p_0\delta_{\tau\tau_1'}\delta(\boldsymbol{p}-\boldsymbol{p}_1')\rho^{s_1's',\tau'}_{s_1s,\tau_1}\left(\begin{array}{c|c} \boldsymbol{k}_1'\ \boldsymbol{k}' & \boldsymbol{p}' \\ \boldsymbol{k}_1\ \boldsymbol{k} & \boldsymbol{p}_1 \end{array}\right)\Bigg].
\end{aligned} \tag{65}$$

In order to proceed further we will use one additional assumption described in the following section. Then we explore once more the smallness of the interaction scales in comparison with the scales of macroscopic changes.

3.4. Approximation of Weakening of Correlations

The matrices depending on variables of a number of particles contain the information about correlations between these particles. In kinetic theory, usually one applies an approximation of molecular chaos (Silin 1971). According to that approximation, there are no correlations before the interaction, i.e.

$$\rho^{s_1',\tau_1'}_{s_1,\tau_1}\left(\begin{array}{c|c|c} \boldsymbol{k}_1' & \boldsymbol{p}_1' & \\ \boldsymbol{k}_1 & \boldsymbol{p}_1 & \end{array} t_0\right) = \rho^{s_1'}_{s_1}\begin{pmatrix} \boldsymbol{k}_1' \\ \boldsymbol{k}_1 \end{pmatrix}\rho^{\tau_1'}_{\tau_1}\begin{pmatrix} \boldsymbol{p}_1' \\ \boldsymbol{p}_1 \end{pmatrix}. \tag{66}$$

An interaction creates the correlations. However, according to the principle of weakening of correlations which is satisfied for sufficiently rarefied gases, the correlations are accounted for only in equation (64) for the one-particle photon matrix via kernel ρ entering the right hand side of the aforementioned equation. This kernel is assumed to characterize the electron and photon states *after* the interaction. It can be represented via the same kernel *before* the interaction and the correlation function as it is done in equation (65).

The two-particle truncated matrices can be presented as products of the one-particle density matrices accounting for the exchange effects (symmetrical and anti-symmetrical forms for photons and electrons, respectively)

$$\rho^{s_1's_2'}_{s_1s_2}\begin{pmatrix} \boldsymbol{k}_1'\ \boldsymbol{k}_2' \\ \boldsymbol{k}_1\ \boldsymbol{k}_2 \end{pmatrix} = \rho^{s_1'}_{s_1}\begin{pmatrix} \boldsymbol{k}_1' \\ \boldsymbol{k}_1 \end{pmatrix}\rho^{s_2'}_{s_2}\begin{pmatrix} \boldsymbol{k}_2' \\ \boldsymbol{k}_2 \end{pmatrix} + \rho^{s_2'}_{s_1}\begin{pmatrix} \boldsymbol{k}_2' \\ \boldsymbol{k}_1 \end{pmatrix}\rho^{s_1'}_{s_2}\begin{pmatrix} \boldsymbol{k}_1' \\ \boldsymbol{k}_2 \end{pmatrix},\quad \rho^{\tau_1'\tau_2'}_{\tau_1\tau_2}\begin{pmatrix} \boldsymbol{p}_1'\ \boldsymbol{p}_2' \\ \boldsymbol{p}_1\ \boldsymbol{p}_2 \end{pmatrix} = \rho^{\tau_1'}_{\tau_1}\begin{pmatrix} \boldsymbol{p}_1' \\ \boldsymbol{p}_1 \end{pmatrix}\rho^{\tau_2'}_{\tau_2}\begin{pmatrix} \boldsymbol{p}_2' \\ \boldsymbol{p}_2 \end{pmatrix} - \rho^{\tau_2'}_{\tau_1}\begin{pmatrix} \boldsymbol{p}_2' \\ \boldsymbol{p}_1 \end{pmatrix}\rho^{\tau_1'}_{\tau_2}\begin{pmatrix} \boldsymbol{p}_1' \\ \boldsymbol{p}_2 \end{pmatrix}. \tag{67}$$

For matrices of higher order entering equation (65), similar equations hold

$$\rho^{s_1's_2',\tau_1'}_{s_1s_2,\tau_1}\left(\begin{array}{c|c} \boldsymbol{k}_1'\ \boldsymbol{k}_2' & \boldsymbol{p}_1' \\ \boldsymbol{k}_1\ \boldsymbol{k}_2 & \boldsymbol{p}_1 \end{array}\right) = \rho^{s_1's_2'}_{s_1s_2}\begin{pmatrix} \boldsymbol{k}_1'\ \boldsymbol{k}_2' \\ \boldsymbol{k}_1\ \boldsymbol{k}_2 \end{pmatrix}\rho^{\tau_1'}_{\tau_1}\begin{pmatrix} \boldsymbol{p}_1' \\ \boldsymbol{p}_1 \end{pmatrix},\qquad \rho^{s_1',\tau_1'\tau_2'}_{s_1,\tau_1\tau_2}\left(\begin{array}{c|c} \boldsymbol{k}_1' & \boldsymbol{p}_1'\ \boldsymbol{p}_2' \\ \boldsymbol{k}_1 & \boldsymbol{p}_1\ \boldsymbol{p}_2 \end{array}\right) = \rho^{s_1'}_{s_1}\begin{pmatrix} \boldsymbol{k}_1' \\ \boldsymbol{k}_1 \end{pmatrix}\rho^{\tau_1'\tau_2'}_{\tau_1\tau_2}\begin{pmatrix} \boldsymbol{p}_1'\ \boldsymbol{p}_2' \\ \boldsymbol{p}_1\ \boldsymbol{p}_2 \end{pmatrix}. \tag{68}$$

It can be shown that these matrices satisfy all normalization conditions.

Now we substitute equations (66)–(68) into the rhs of equation (65), and the resulting expression into equation (64). At this step, after all substitutions, we will use the fact that the matrices in the rhs of equation (65) do not depend on time during the interaction. Then we can take the integrals over the space-time variables enter the matrices $\mathcal{N}$ only. Simultaneously we let $V\to\infty$, $t_0\to-\infty$, keeping the upper limit t finite. As a result of this procedure four integrals over space give four three-dimensional δ-functions as in (55), namely two pairs of the type

$$\delta(\boldsymbol{p}'+\boldsymbol{k}'-\boldsymbol{p}_{\mathrm{v}})\delta(\boldsymbol{p}+\boldsymbol{k}-\boldsymbol{p}_{\mathrm{v}}) = \delta(\boldsymbol{p}'+\boldsymbol{k}'-\boldsymbol{p}-\boldsymbol{k})\delta(\boldsymbol{p}+\boldsymbol{k}-\boldsymbol{p}_{\mathrm{v}}) \tag{69}$$

and

$$\delta(\boldsymbol{p}'-\boldsymbol{k}-\boldsymbol{p}_{\mathrm{v}})\delta(\boldsymbol{p}-\boldsymbol{k}'-\boldsymbol{p}_{\mathrm{v}}) = \delta(\boldsymbol{p}'+\boldsymbol{k}'-\boldsymbol{p}-\boldsymbol{k})\delta(\boldsymbol{p}-\boldsymbol{k}'-\boldsymbol{p}_{\mathrm{v}}). \tag{70}$$

Both combinations give the δ-functions reflecting the momentum conservation. The second set of δ-functions gives a possibility to take the integrals over $\boldsymbol{p}_{\mathrm{v}}$.

Integrals over time with the upper limit t can be taken independently. They have the form

$$\int\limits_{-\infty}^{t} e^{i(k'+p_0'-p_{\mathrm{v}}^0)ct'/\hbar}\mathrm{d}t' = e^{i(k'+p_0'-p_{\mathrm{v}}^0)ct/\hbar}\left[\pi\hbar\delta(k'+p_0'-p_{\mathrm{v}}^0) - \frac{i\hbar}{k'+p_0'-p_{\mathrm{v}}^0}\right]. \tag{71}$$

Integrals over $p_{\rm v}^0$ with the second terms of this type must be calculated as the Cauchy principal values. These terms do not ensure the energy conservation law. They vanish if the limit of integration t is equal to ∞.

The terms with absence of energy conservation in kinetic equations, as was noticed by Nagirner (1994), appear in Silin (1974) and Bomier (1991) who do not discuss this fact. In their papers, the interaction between electrons (Silin 1971) and between photons and atoms (Bomier 1991) does not depend on time. In our case, the Hamiltonian is more complicated and without the energy conservation law we cannot calculate the scattering amplitudes and the cross–sections of the process.

We suppose that such terms are not physical and must be excluded. The interaction between photon and electron has finished before the density matrix changes noticeably and the integration limit t must be taken infinite. Then the resulting integral is given by equation (55). However, this procedure gives the cross-section twice as large as the correct value. To overcome this difficulty we continue the upper limit t to ∞ but take half of the result.

The next step is to use for the last time the assumption of the small scale of the interaction. Instead of the one-particle kernels we use now their diagonal in momentum forms (39) and (41). Accounting for relation (49), the lhs of equation (64) then takes the form

$$T_0\frac{\mathrm{d}}{\mathrm{d}t}\rho_{s_1}^{s_1'}\left(\begin{matrix}\boldsymbol{k}_1'\\ \boldsymbol{k}_1\end{matrix}\middle| t\right) = T_0 k_1\delta(\boldsymbol{k}_1'-\boldsymbol{k}_1)\frac{\mathrm{d}}{\mathrm{d}t}\rho_{s_1}^{s_1'}(\boldsymbol{k}_1) = k_1\delta(\underline{k}_1'-\underline{k}_1)\frac{2\pi\hbar}{c}\frac{\mathrm{d}}{\mathrm{d}t}\rho_{s_1}^{s_1'}(\boldsymbol{k}_1). \tag{72}$$

After all the substitutions mentioned above, the rhs of equation (64) contains 22 terms. The two terms are the result of the substitution of (66). All other terms in equations (64) and (65) come in couples, so that the total number of terms is multiple of 4. Every term in the first line in the square brackets of (65) gives two terms, and every term from other two pairs of terms gives four terms, resulting in 20 terms.

The first two terms containing linearly the amplitude M given by equation (56) cancel out. This follows from the equality

$$M_{s'\tau'}^{s\ \tau}\left(\begin{matrix}\boldsymbol{k}\\ \boldsymbol{k}\end{matrix}\middle|\begin{matrix}\boldsymbol{p}\\ \boldsymbol{p}\end{matrix}\right) = \delta_{ss'}\delta_{\tau\tau'}, \tag{73}$$

which can be easily proven using properties of the solutions of the Dirac equation. Equality (73) leads also to the cancellation of the eight terms arising from the first terms in the rhs of equations (67).

All the remaining twelve terms contain eight three-dimensional integrals over photon and electron momenta. There are two four-dimensional and five three-dimensional δ-functions in the integrand. All three-dimensional δ-functions disappear when taking the integrals. The resulting expression contains three three-dimensional integrals (one over photon momenta and two over electron momenta), and two four-dimensional δ-functions, $\delta(\underline{p}'+\underline{k}'-\underline{p}-\underline{k})$ and $\delta(\underline{k}_1'-\underline{k}_1)$. The first one reflects the energy and momentum conservation laws. The second one cancels out since it also appears in the lhs of equation (64) according to equation (72).

In the final expression, four out of twelve terms contain products of one photon and one electron matrices corresponding to the spontaneous scattering. Two of these terms entering with the plus sign are responsible for the emission and the two terms entering with the minus sign are responsible for the attenuation of the photon beam due to scattering. The remaining eight terms reflect the exchange effects. Four terms containing the product of one electron and two photon matrices correspond to the induced scattering, and four terms containing the product of one photon and two electron matrices reflect the exclusion principle. The induced scattering terms have the same signs as the spontaneous terms, while the exclusion principle terms have the opposite signs. After an elementary but very long deduction, we obtain the sought kinetic equation.

4. Kinetic Equations for Photons and Electrons

4.1. Kinetic Equation for Polarized Photons

As we mentioned above, all the remaining terms in the rhs of equation (64) after substitution of equation (66) contain the factor $\delta(\underline{k}_1'-\underline{k}_1)$ which also exists in the lhs of the equation. This factor therefore cancels out. We can now restore the dependence of the photon matrix on time and spatial coordinates. Let us also change the designations, writing the photon matrix in the lhs in the form $\rho_{s_0}^{s_0'}(\boldsymbol{k})$. Rewriting the derivative over the line of sight as the full derivative, the lhs of the equation takes the covariant form

$$\underline{k}\ \underline{\nabla}\rho_{s_0}^{s_0'}(\boldsymbol{k},\boldsymbol{r},t) = \left(\frac{k}{c}\frac{\partial}{\partial t}+\boldsymbol{k}\cdot\boldsymbol{\nabla}\right)\rho_{s_0}^{s_0'}(\boldsymbol{k},\boldsymbol{r},t). \tag{74}$$

In the rhs of the equation we change the indices and add and subtract two pairs of identical terms containing products of two photon and two electron matrices for symmetry. The resulting kinetic equation takes the form

$$
\begin{aligned}
\underline{k}\,\underline{\nabla}\rho^{s'_0}_{s_0}(\boldsymbol{k},\boldsymbol{r},t) &= -\frac{r_0^2}{2}\frac{m^2c^2}{(2\pi\hbar)^3}\int\frac{\mathrm{d}^3k'}{k'}\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p'}{p'_0}\delta(\underline{p}'+\underline{k}'-\underline{p}-\underline{k})\\
&\times\Big\{\rho^{s''}_{s_0}(\boldsymbol{k})\rho^{\tau''}_{\tau'''}(\boldsymbol{p})M^{s\,\tau}_{s''\tau''}\left(\begin{array}{c|c}\boldsymbol{k}'&\boldsymbol{p}'\\\boldsymbol{k}&\boldsymbol{p}\end{array}\right)\left[\delta^{s'}_{s}+\rho^{s'}_{s}(\boldsymbol{k}')\right]\left[\delta^{\tau'}_{\tau}-\rho^{\tau'}_{\tau}(\boldsymbol{p}')\right]M^{s'_0\tau'''}_{s'\,\tau'}\left(\begin{array}{c|c}\boldsymbol{k}&\boldsymbol{p}\\\boldsymbol{k}'&\boldsymbol{p}'\end{array}\right)\\
&+M^{s''\tau''}_{s_0\tau'''}\left(\begin{array}{c|c}\boldsymbol{k}'&\boldsymbol{p}'\\\boldsymbol{k}&\boldsymbol{p}\end{array}\right)\left[\delta^{s'}_{s''}+\rho^{s'}_{s''}(\boldsymbol{k}')\right]\left[\delta^{\tau'}_{\tau''}-\rho^{\tau'}_{\tau''}(\boldsymbol{p}')\right]M^{s\,\tau}_{s'\tau'}\left(\begin{array}{c|c}\boldsymbol{k}&\boldsymbol{p}\\\boldsymbol{k}'&\boldsymbol{p}'\end{array}\right)\rho^{s'_0}_{s}(\boldsymbol{k})\rho^{\tau'''}_{\tau}(\boldsymbol{p})\\
&-M^{s''\tau''}_{s_0\tau'''}\left(\begin{array}{c|c}\boldsymbol{k}'&\boldsymbol{p}'\\\boldsymbol{k}&\boldsymbol{p}\end{array}\right)\rho^{s'}_{s''}(\boldsymbol{k}')\rho^{\tau'}_{\tau''}(\boldsymbol{p}')M^{s\,\tau}_{s'\tau'}\left(\begin{array}{c|c}\boldsymbol{k}&\boldsymbol{p}\\\boldsymbol{k}'&\boldsymbol{p}'\end{array}\right)\left[\delta^{s'_0}_{s}+\rho^{s'_0}_{s}(\boldsymbol{k})\right]\left[\delta^{\tau'''}_{\tau}-\rho^{\tau'''}_{\tau}(\boldsymbol{p})\right]\\
&-\left[\delta^{s''}_{s_0}+\rho^{s''}_{s_0}(\boldsymbol{k})\right]\left[\delta^{\tau''}_{\tau'''}-\rho^{\tau''}_{\tau'''}(\boldsymbol{p})\right]M^{s\,\tau}_{s''\tau''}\left(\begin{array}{c|c}\boldsymbol{k}'&\boldsymbol{p}'\\\boldsymbol{k}&\boldsymbol{p}\end{array}\right)\rho^{s'}_{s}(\boldsymbol{k}')\rho^{\tau'}_{\tau}(\boldsymbol{p}')M^{s'_0\tau'''}_{s'\,\tau'}\left(\begin{array}{c|c}\boldsymbol{k}&\boldsymbol{p}\\\boldsymbol{k}'&\boldsymbol{p}'\end{array}\right)\Big\}\,.
\end{aligned}\tag{75}
$$

Here $r_0=e^2/mc^2$ is the classical electron radius. The arguments $\boldsymbol{r}$ and t of the matrices in the collision integral are omitted. The terms in the first two lines in braces describe the photons outgoing from the state with momentum $\boldsymbol{k}$, the other two lines describe the photons incoming into this state. In order to outline the matrix character of Kronecker symbol we use further the notation $\delta^{s'}_{s}$, where the lower index represents the rows and the upper index represents the columns.

Note that the products of four matrices are added just for the symmetry of the equation. If these four terms are dropped and we change some indices, the equation takes the form

$$
\begin{aligned}
\underline{k}\,\underline{\nabla}\rho^{s'_0}_{s_0}(\boldsymbol{k},\boldsymbol{r},t) &= -\frac{r_0^2}{2}\frac{m^2c^2}{(2\pi\hbar)^3}\int\frac{\mathrm{d}^3k'}{k'}\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p'}{p'_0}\delta(\underline{p}'+\underline{k}'-\underline{p}-\underline{k})\\
&\times\Big\{\rho^{s''}_{s_0}(\boldsymbol{k})\rho^{\tau''}_{\tau'''}(\boldsymbol{p})\left[M^{s'\,\tau'}_{s''\tau''}-M^{s'\,\tau}_{s''\tau''}\rho^{\tau'}_{\tau}(\boldsymbol{p}')+M^{s\,\tau'}_{s''\tau''}\rho^{s'}_{s}(\boldsymbol{k}')\right]M^{s'_0\tau'''}_{s'\,\tau'}\\
&+M^{s''\tau''}_{s_0\,\tau'''}\left[M^{s\,\tau}_{s''\tau''}-\rho^{\tau'}_{\tau''}(\boldsymbol{p}')M^{s\,\tau}_{s''\tau'}+\rho^{s'}_{s''}(\boldsymbol{k}')M^{s\,\tau}_{s'\tau''}\right]\rho^{s'_0}_{s}(\boldsymbol{k})\rho^{\tau'''}_{\tau}(\boldsymbol{p})\\
&-2M^{s\,\tau}_{s_0\tau''}\rho^{s'}_{s}(\boldsymbol{k}')\rho^{\tau'}_{\tau}(\boldsymbol{p}')M^{s'_0\tau''}_{s'\,\tau'}+2M^{s\,\tau}_{s_0\tau''}\rho^{s'}_{s}(\boldsymbol{k}')\rho^{\tau'}_{\tau}(\boldsymbol{p}')M^{s'_0\tau'''}_{s'\,\tau'}\rho^{\tau''}_{\tau'''}(\boldsymbol{p})\\
&-M^{s\,\tau}_{s_0\tau''}\rho^{s'}_{s}(\boldsymbol{k}')\rho^{\tau'}_{\tau}(\boldsymbol{p}')M^{s''\tau''}_{s'\,\tau'}\rho^{s'_0}_{s''}(\boldsymbol{k})-\rho^{s''}_{s_0}(\boldsymbol{k})M^{s\,\tau}_{s''\tau''}\rho^{s'}_{s}(\boldsymbol{k}')\rho^{\tau'}_{\tau}(\boldsymbol{p}')M^{s'_0\tau''}_{s'\,\tau'}\Big\}\,.
\end{aligned}\tag{76}
$$

In the last equation we omit the arguments of scattering amplitudes M for brevity. These arguments are the same and in the same order as in equation (75).

We note, that a possibility to calculate the scattering amplitudes appears after we choose the polarization basis for photons and the axis where one projects the electron spin. As the axis one can use the electron momenta. The situation is more complex for the polarization bases. We will give detailed description of that later and now turn to the equation for the electron distribution function.

4.2. Kinetic Equation for Polarized Electrons

Now we also can write down the kinetic equation for electrons. It can be deduced following the same procedure as in the case of the photon equation. However, due to the symmetry of equation (75), we can write this equation by analogy. One just removes one integral over electron momentum and adds one integral over photon momentum as well as sums over photon polarization and removes summation over indices of electron polarization (with the external momentum, $\boldsymbol{p}$). The resulting relativistic kinetic equation takes the form

$$
\begin{aligned}
\underline{p}\,\underline{\nabla}\rho^{\tau'_0}_{\tau_0}(\boldsymbol{p},\boldsymbol{r},t) &= -\frac{r_0^2}{2}\frac{m^2c^2}{(2\pi\hbar)^3}\int\frac{\mathrm{d}^3k}{k}\frac{\mathrm{d}^3k'}{k'}\frac{\mathrm{d}^3p'}{p'_0}\delta(\underline{p}'+\underline{k}'-\underline{p}-\underline{k})\\
&\times\Big\{\rho^{s''}_{s'''}(\boldsymbol{k})\rho^{\tau''}_{\tau_0}(\boldsymbol{p})M^{s\,\tau}_{s''\tau''}\left(\begin{array}{c|c}\boldsymbol{k}'&\boldsymbol{p}'\\\boldsymbol{k}&\boldsymbol{p}\end{array}\right)\left[\delta^{s'}_{s}+\rho^{s'}_{s}(\boldsymbol{k}')\right]\left[\delta^{\tau'}_{\tau}-\rho^{\tau'}_{\tau}(\boldsymbol{p}')\right]M^{s'''\tau'_0}_{s'\,\tau'}\left(\begin{array}{c|c}\boldsymbol{k}&\boldsymbol{p}\\\boldsymbol{k}'&\boldsymbol{p}'\end{array}\right)\\
&+M^{s''\tau''}_{s'''\tau_0}\left(\begin{array}{c|c}\boldsymbol{k}'&\boldsymbol{p}'\\\boldsymbol{k}&\boldsymbol{p}\end{array}\right)\left[\delta^{s'}_{s''}+\rho^{s'}_{s''}(\boldsymbol{k}')\right]\left[\delta^{\tau'}_{\tau''}-\rho^{\tau'}_{\tau''}(\boldsymbol{p}')\right]M^{s\,\tau}_{s'\tau'}\left(\begin{array}{c|c}\boldsymbol{k}&\boldsymbol{p}\\\boldsymbol{k}'&\boldsymbol{p}'\end{array}\right)\rho^{s'_0}_{s}(\boldsymbol{k})\rho^{\tau'_0}_{\tau}(\boldsymbol{p})\\
&-M^{s''\tau''}_{s'''\tau_0}\left(\begin{array}{c|c}\boldsymbol{k}'&\boldsymbol{p}'\\\boldsymbol{k}&\boldsymbol{p}\end{array}\right)\rho^{s'}_{s''}(\boldsymbol{k}')\rho^{\tau'}_{\tau''}(\boldsymbol{p}')M^{s\,\tau}_{s'\tau'}\left(\begin{array}{c|c}\boldsymbol{k}&\boldsymbol{p}\\\boldsymbol{k}'&\boldsymbol{p}'\end{array}\right)\left[\delta^{s'''}_{s}+\rho^{s'''}_{s}(\boldsymbol{k})\right]\left[\delta^{\tau'_0}_{\tau}-\rho^{\tau'_0}_{\tau}(\boldsymbol{p})\right]\\
&-\left[\delta^{s''}_{s'''}+\rho^{s''}_{s'''}(\boldsymbol{k})\right]\left[\delta^{\tau''}_{\tau'_0}-\rho^{\tau''}_{\tau'_0}(\boldsymbol{p})\right]M^{s\,\tau}_{s''\tau''}\left(\begin{array}{c|c}\boldsymbol{k}'&\boldsymbol{p}'\\\boldsymbol{k}&\boldsymbol{p}\end{array}\right)\rho^{s'}_{s}(\boldsymbol{k}')\rho^{\tau'}_{\tau}(\boldsymbol{p}')M^{s'''\tau'_0}_{s'\,\tau'}\left(\begin{array}{c|c}\boldsymbol{k}&\boldsymbol{p}\\\boldsymbol{k}'&\boldsymbol{p}'\end{array}\right)\Big\}\,.
\end{aligned}\tag{77}
$$

It is easy to check that in the arbitrary time moment, t, and in the arbitrary point in space, $\boldsymbol{r}$, two relations are valid

$$\underline{\nabla}\int \underline{k}\,\rho_{s_0}^{s_0}(\boldsymbol{k},\boldsymbol{r},t)\frac{\mathrm{d}^3k}{k}=0,\quad \underline{\nabla}\int \underline{p}\,\rho_{\tau_0}^{\tau_0}(\boldsymbol{p},\boldsymbol{r},t)\frac{\mathrm{d}^3p}{p_0}=0. \tag{78}$$

These relations reflect the continuity equations for the photon and electron number and are the consequence of the conservation of the number and quality of particles during the scattering. It is also easy to check the validity of the energy-momentum conservation for Compton scattering

$$\nabla_\mu\left[\int k^\mu k^\nu \rho_{s_0}^{s_0}(\boldsymbol{k},\boldsymbol{r},t)\frac{\mathrm{d}^3k}{k}+\int p^\mu p^\nu \rho_{\tau_0}^{\tau_0}(\boldsymbol{p},\boldsymbol{r},t)\frac{\mathrm{d}^3p}{p_0}\right]=0. \tag{79}$$

We note that conservation laws (78) and (79) are valid only for the occupation numbers, but not for the polarizations.

4.3. Scattering Amplitudes

Matrix elements (56) for Compton scattering (amplitudes) can be rather easily calculated for a special choice of polarization bases — internal bases. We take the unit vectors of these bases in the following form. For the photon of momentum $\boldsymbol{k}$

$$\underline{e}_{(1)}^{\rm in}(\boldsymbol{k})=\frac{\xi'\underline{k}+\xi\underline{k}'-q\underline{p}}{m\,c\,q\,\Delta},\quad \underline{e}_{(2)}^{\rm in}(\boldsymbol{k})=\frac{1}{m^3\,c^3\,q\,\Delta}\left\{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{p},p_0\boldsymbol{k}\times\boldsymbol{k}'+k\boldsymbol{k}'\times\boldsymbol{p}+k'\boldsymbol{p}\times\boldsymbol{k}\right\}. \tag{80}$$

Here the dimensionless scalar products

$$\xi\equiv\underline{k}\;\underline{p}/m^2c^2=\underline{k}'\;\underline{p}'/m^2c^2,\quad \xi'\equiv\underline{k}'\;\underline{p}/m^2c^2=\underline{k}\;\underline{p}'/m^2c^2,\quad q\equiv\underline{k}\;\underline{k}'/m^2c^2, \tag{81}$$

and $\Delta=\sqrt{2\xi\xi'/q-1}$. The double definitions of ξ and ξ' in equation (81) as well as the relation $\xi=\xi'+q$ are the consequence of the conservation laws.

The values ξ and ξ' are the photon energies before and after scattering in the frame where the electron before interaction is at rest. If we denote the cosine of scattering angle in this frame as μ_0 then

$$q=\xi\xi'(1-\mu_0),\quad \xi'=\frac{\xi}{1+\xi(1-\mu_0)},\quad \Delta=\sqrt{\frac{1+\mu_0}{1-\mu_0}},\quad \mu_0=1+\frac{1}{\xi}-\frac{1}{\xi'}. \tag{82}$$

For the photon of momentum $\boldsymbol{k}'$, the unit vectors are very similar

$$\underline{e}_{(1)}^{\rm in}(\boldsymbol{k}')=\underline{e}_{(1)}^{\rm in}(\boldsymbol{k}),\quad \underline{e}_{(2)}^{\rm in}(\boldsymbol{k}')=-\underline{e}_{(2)}^{\rm in}(\boldsymbol{k}). \tag{83}$$

One can add a vector of the photon four-momentum $\underline{k}$ (or $\underline{k}'$) multiplied by any real number to the polarization unit vectors. By doing so, one can make the time components of these unit vectors (as well as of the external vectors $\underline{e}(\boldsymbol{k})$) equal to zero. For such vectors, the scattering amplitudes (matrix elements) which are marked with a circle atop of the letter, are

$$\begin{aligned}
\overset{\circ}{M}{}_{11}^{11} &= \overset{\circ}{M}{}_{12}^{12}=\frac{1}{\Gamma}U_+\left(V_+-\frac{2}{\Gamma}V_-\right), & \overset{\circ}{M}{}_{11}^{12} &= -\overset{\circ}{M}{}_{12}^{11}=\frac{1}{\Gamma}U_-\left(V_--\frac{2}{\Gamma}V_+\right), & & \\
\overset{\circ}{M}{}_{11}^{21} &= -\overset{\circ}{M}{}_{12}^{22}=i\frac{\xi}{\Gamma^3}(U_-V_+-U_+W_+), & \overset{\circ}{M}{}_{11}^{22} &= \overset{\circ}{M}{}_{12}^{21}=i\frac{\xi}{\Gamma^3}(U_+V_--U_-W_-), & \overset{\circ}{M}{}_{21}^{21} &= \overset{\circ}{M}{}_{22}^{22}=\frac{1}{\Gamma}U_+V_+,\\
\overset{\circ}{M}{}_{21}^{11} &= -\overset{\circ}{M}{}_{22}^{12}=i\frac{\xi'}{\Gamma^3}(U_-V_++U_+W_+), & \overset{\circ}{M}{}_{21}^{12} &= \overset{\circ}{M}{}_{22}^{11}=i\frac{\xi'}{\Gamma^3}(U_+V_-+U_-W_-), & \overset{\circ}{M}{}_{21}^{22} &= -\overset{\circ}{M}{}_{22}^{21}=\frac{1}{\Gamma}U_-V_-.
\end{aligned} \tag{84}$$

Here we denoted

$$\begin{aligned}
V_\pm &= \sqrt{\Gamma+1}\sqrt{\Gamma'+1}\pm\sqrt{\Gamma-1}\sqrt{\Gamma'-1},\quad W_\pm=\sqrt{\Gamma+1}\sqrt{\Gamma'-1}\pm\sqrt{\Gamma-1}\sqrt{\Gamma'+1},\\
U_\pm &= \frac{1}{4}\frac{\Delta'+Z\pm\Delta}{\sqrt{\Delta'(\Delta'+Z)}},\quad \Gamma=\sqrt{\Delta^2+1}=\sqrt{\frac{2\xi\xi'}{q}}=\sqrt{\frac{2}{1-\mu_0}},\\
\Gamma' &= \frac{\Gamma}{2}\left(\frac{\xi}{\xi'}+\frac{\xi'}{\xi}\right)=\frac{\Gamma^2+q}{\Gamma},\quad Z=\frac{\Gamma}{2}q\left(\frac{1}{\xi'}+\frac{1}{\xi}\right)=\frac{\xi+\xi'}{\Gamma},\quad \Delta'=\sqrt{(\Gamma')^2-1}=\sqrt{\Delta^2+Z^2}.
\end{aligned} \tag{85}$$

Note that the scattering amplitudes enter equations (75) and (77) without summation on their indices only with the products of four density matrices (two photon and two electron ones), but as we mentioned these products annihilate.

In all other terms, as we can see in the equation for photons in the form (76), there are summations over at least one pair of indices of scattering amplitudes.

If indeed the summation is over only one pair of indices then the resulting sum depends on the 6 remaining indices and contains $2^6 = 64$ terms altogether. If, on the other hand, the summation is fulfilled over two pairs of indices, then the sum contains $2^4 = 16$ terms. The term which describes the attenuation of radiation belongs to the last category

$$t^{s\,\tau}_{s'\tau'} = M^{s''\tau''}_{s'\,\tau'}\begin{pmatrix}\boldsymbol{k}' & \boldsymbol{p}'\\ \boldsymbol{k} & \boldsymbol{p}\end{pmatrix} M^{s\,\tau}_{s''\tau''}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p}\\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix}, \tag{86}$$

where the sum is taken over one pair of photon indices and over one pair of electron indices. Here one gets

$$\overset{\circ}{t}{}^{12}_{11} = \overset{\circ}{t}{}^{11}_{12} = \overset{\circ}{t}{}^{22}_{21} = \overset{\circ}{t}{}^{21}_{22} = 0, \qquad \overset{\circ}{t}{}^{11}_{11} = \overset{\circ}{t}{}^{12}_{12} = \frac{B}{2} - 1 + \mu_0^2, \quad \overset{\circ}{t}{}^{21}_{21} = \overset{\circ}{t}{}^{22}_{22} = \frac{B}{2},$$
$$\overset{\circ}{t}{}^{22}_{11} = \overset{\circ}{t}{}^{21}_{12} = -\overset{\circ}{t}{}^{12}_{21} = -\overset{\circ}{t}{}^{11}_{22} = i\frac{q}{\Gamma^3}\left(1 - \frac{1}{\xi'}\right), \; \overset{\circ}{t}{}^{21}_{11} = \overset{\circ}{t}{}^{12}_{22} = -\overset{\circ}{t}{}^{22}_{12} = -\overset{\circ}{t}{}^{11}_{21} = i\frac{\Delta}{\Gamma^3}(\xi' + \xi\mu_0), \tag{87}$$

where $B = \xi/\xi' + \xi'/\xi$.

4.4. Transformation Matrices

We have deduced the formulae for the Compton scattering amplitudes (84) in a specially chosen internal polarization basis (80). However, the expressions (54) and (56) for these amplitudes contain unit vectors of some external bases which should reflect the geometry of the scattering medium. The photon density matrices entering the kinetic equations (75)-(77) are connected to the same (external) vectors. Therefore, we must find transformations between the external and the internal bases.

Two sets of polarization vectors $\underline{e}_{(s)}(\boldsymbol{k})$ and $\underline{e}^{\rm in}_{(r)}(\boldsymbol{k})$ together with the vector $\boldsymbol{k}/k$ form two three-dimensional bases with one common unit vector. Consequently, the external vectors and the internal vectors are connected to one another via relation

$$\underline{e}_{(s)}(\boldsymbol{k}) = l^{\cdot\, r}_{s\,\cdot}(-\chi)\underline{e}^{\rm in}_{(r)}(\boldsymbol{k}), \tag{88}$$

where the angle χ is defined in the following manner

$$\cos\chi = -\underline{e}_{(1)}(\boldsymbol{k})\underline{e}^{\rm in}_{(1)}(\boldsymbol{k}) = -\underline{e}_{(2)}(\boldsymbol{k})\underline{e}^{\rm in}_{(2)}(\boldsymbol{k}), \quad \sin\chi = \underline{e}_{(1)}(\boldsymbol{k})\underline{e}^{\rm in}_{(2)}(\boldsymbol{k}) = -\underline{e}_{(2)}(\boldsymbol{k})\underline{e}^{\rm in}_{(1)}(\boldsymbol{k}). \tag{89}$$

The specific expressions for $\cos\chi$ and $\sin\chi$ are given in Nagirner & Poutanen (1993) where the external vectors are defined for the plane-parallel medium. The transformation matrix

$$\{l^{\cdot\, r}_{s\,\cdot}(\chi)\} = \begin{pmatrix}\cos\chi & \sin\chi\\ -\sin\chi & \cos\chi\end{pmatrix} \tag{90}$$

has the following properties

$$l^{\cdot\, r}_{s\,\cdot}(\chi) = l^{s}_{\cdot\, r}(\chi) = l^{\cdot\, s}_{r\,\cdot}(-\chi) = l^{r}_{\cdot\, s}(-\chi). \tag{91}$$

These properties are obvious if the numbers 1, 2 are substituted instead of indices. Matrices of this kind are commutative and multiplicative, and their transposition is equivalent to the inversion or changing the sign of the argument.

Multiplying both sides of equations (88) by the Dirac matrices we get

$$\hat{e}_{(s)}(\boldsymbol{k}) = l^{\cdot\, r}_{s\,\cdot}(-\chi)\hat{e}^{\rm in}_{(r)}(\boldsymbol{k}), \; \hat{e}_{(s')}(\boldsymbol{k}') = l^{\cdot\, r'}_{s'\,\cdot}(-\chi')\hat{e}^{\rm in}_{(r')}(\boldsymbol{k}'). \tag{92}$$

The transformations of bases related to the photon momentum $\boldsymbol{k}'$ are analogous to (88) and the angle χ' is defined analogously to χ in equation (89). Then, for amplitude (56), we have

$$\begin{aligned} M^{s\,\tau}_{s'\tau'}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p}\\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix} &= \frac{1}{2}\overline{u}^{\tau'}(\boldsymbol{p}')\left[\hat{e}_{(s)}(\boldsymbol{k})\frac{mc + \hat{p} - \hat{k}'}{mc\xi'}\hat{e}_{(s')}(\boldsymbol{k}') - \hat{e}_{(s')}(\boldsymbol{k}')\frac{mc + \hat{p} + \hat{k}}{mc\xi}\hat{e}_{(s)}(\boldsymbol{k})\right]u^{\tau}(\boldsymbol{p}) \\ &= l^{\cdot\, r}_{s\,\cdot}(-\chi)l^{\cdot\, r'}_{s'\,\cdot}(-\chi')\frac{1}{2}\overline{u}^{\tau'}(\boldsymbol{p}')\left[\hat{e}^{\rm in}_{(r)}(\boldsymbol{k})\frac{mc + \hat{p} - \hat{k}'}{mc\xi'}\hat{e}^{\rm in}_{(r')}(\boldsymbol{k}') - \hat{e}^{\rm in}_{(r')}(\boldsymbol{k}')\frac{mc + \hat{p} + \hat{k}}{mc\xi}\hat{e}^{\rm in}_{(r)}(\boldsymbol{k})\right]u^{\tau}(\boldsymbol{p}) \\ &= l^{\cdot\, r}_{s\,\cdot}(-\chi)l^{\cdot\, r'}_{s'\,\cdot}(-\chi')\overset{\circ}{M}{}^{r\,\tau}_{r'\tau'}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p}\\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix}. \end{aligned} \tag{93}$$

Using equations (57) and (93) we obtain the following expression for the matrix element with the primed upper momenta

$$M^{s\,\tau}_{s'\tau'}\begin{pmatrix}\boldsymbol{k}' & \boldsymbol{p}'\\ \boldsymbol{k} & \boldsymbol{p}\end{pmatrix} = \left[M^{s'\tau'}_{s\,\tau}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p}\\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix}\right]^* = \left[l^{s'}_{\cdot\, r'}(-\chi)l^{\cdot\, r}_{s\,\cdot}(-\chi')\overset{\circ}{M}{}^{r'\tau'}_{r\,\tau}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p}\\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix}\right]^* = l^{s'}_{\cdot\, r'}(-\chi)l^{\cdot\, r}_{s\,\cdot}(-\chi')\left[\overset{\circ}{M}{}^{r\,\tau}_{r'\tau'}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p}\\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix}\right]^*. \tag{94}$$

These formulae must be substituted into equations (75) and (77).

4.5. Equation for Photons Interacting with Unpolarized Electrons

If electrons are unpolarized (i.e., equation [46] is valid), then the matrix $\rho_\tau^{\tau'}(\boldsymbol{p})$ must be replaced by the product $\delta_\tau^{\tau'}(2\pi\hbar)^3 f_{\rm e}(\boldsymbol{p})/2$. The factor $\delta_\tau^{\tau'} - \rho_\tau^{\tau'}(\boldsymbol{p})$ is replaced by $\delta_\tau^{\tau'}[1-(2\pi\hbar)^3 f_{\rm e}(\boldsymbol{p})/2]$, and one can sum over the electron polarizations. For the external bases,

$$T_{s's'''}^{s\,s''}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p} \\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix} = M_{s'\tau'}^{s\,\tau}\begin{pmatrix}\boldsymbol{k}' & \boldsymbol{p}' \\ \boldsymbol{k} & \boldsymbol{p}\end{pmatrix} M_{s'''\tau}^{s''\,\tau'}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p} \\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix} = \left[M_{s\,\tau}^{s'\,\tau'}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p} \\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix}\right]^* M_{s'''\tau}^{s''\,\tau'}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p} \\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix}. \tag{95}$$

Here, one sums over the repeated indices $\tau,\tau' = 1,2$. After summation over electron polarization the Compton scattering cross-sections of polarized radiation cannot be represented as a product anymore. This reflects the transition from the "pure" scattering of photons and electrons with fixed polarization states to the mixed one.

The expressions for the cross–sections (95) have the simplest form in the internal polarization bases (80) and (83). The matrix elements are then given by (84) and therefore

$$\overset{\circ}{T}{}_{s's'''}^{s\,s''}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p} \\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix} = \overset{\circ}{M}{}_{s'\tau'}^{s\,\tau}\begin{pmatrix}\boldsymbol{k}' & \boldsymbol{p}' \\ \boldsymbol{k} & \boldsymbol{p}\end{pmatrix} \overset{\circ}{M}{}_{s'''\tau}^{s''\,\tau'}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p} \\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix} = \left[\overset{\circ}{M}{}_{s\,\tau}^{s'\,\tau'}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p} \\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix}\right]^* \overset{\circ}{M}{}_{s'''\tau}^{s''\,\tau'}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p} \\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix}, \tag{96}$$

where the sums are taken over the repeated indices $\tau,\tau' = 1,2$.

Only 8 out of 16 elements of matrix (96) are not zeros, and only 5 elements are different

$$\begin{aligned}\overset{\circ}{T}{}_{11}^{11} &= \frac{1}{2}[B+2+4(\mu_0^2-1)], \quad \overset{\circ}{T}{}_{22}^{22} = \frac{1}{2}(B+2), \quad \overset{\circ}{T}{}_{21}^{12} = \overset{\circ}{T}{}_{12}^{21} = \frac{1}{2}(B-2), \\ \overset{\circ}{T}{}_{12}^{12} &= \overset{\circ}{T}{}_{21}^{21} = \frac{1}{2}(B+2)\mu_0, \quad \overset{\circ}{T}{}_{11}^{22} = \overset{\circ}{T}{}_{22}^{11} = -\frac{1}{2}(B-2)\mu_0.\end{aligned} \tag{97}$$

The same expressions are obtained if they are derived directly for unpolarized electrons by means of traces and projection operators (see e.g. Berestetskii et al. 1982).

Combination (93) and (94) with (95) gives the law of transformation of the cross–sections of polarized radiation by unpolarized electrons. One can write a transformation, for example, in the following way

$$T_{s's'''}^{s\,s''}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p} \\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix} = l_{s'\cdot}^{\cdot r'}(-\chi)l_{\cdot r}^{s\cdot}(-\chi')\overset{\circ}{T}{}_{r'r'''}^{r\,r''}\begin{pmatrix}\boldsymbol{k} & \boldsymbol{p} \\ \boldsymbol{k}' & \boldsymbol{p}'\end{pmatrix} l_{r''\cdot}^{\cdot s''}(\chi)l_{\cdot s'''}^{r'''\cdot}(\chi'). \tag{98}$$

The transformation matrices can be written in various forms using equation (91).

Keeping in mind the formulae expressing transformation laws and omitting the arguments of the T-matrices, we rewrite the kinetic equation for photons with unpolarized electrons in the form

$$\begin{aligned}\underline{k}\,\underline{\nabla}\rho_{s_0}^{s_0'}(\boldsymbol{k},\boldsymbol{r},t) \;=\;& \frac{r_0^2}{2}\frac{m^2c^2}{2}\int\frac{\mathrm{d}^3k'}{k'}\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p'}{p_0'}\delta(\underline{p}'+\underline{k}'-\underline{p}-\underline{k}) \\ & \Big\{f_{\rm e}(\boldsymbol{p}')\left[1-(2\pi\hbar)^3 f_{\rm e}(\boldsymbol{p})/2\right]2T_{s_0 s'}^{s\;s_0'}\rho_s^{s'}(\boldsymbol{k}') - f_{\rm e}(\boldsymbol{p})\left[1-(2\pi\hbar)^3 f_{\rm e}(\boldsymbol{p}')/2\right]\left[\rho_{s_0}^{s''}(\boldsymbol{k})T_{s''s'}^{s'\;s_0'} + T_{s_0\,s''}^{s''s}\rho_s^{s_0'}(\boldsymbol{k})\right] \\ +\;& \left[f_{\rm e}(\boldsymbol{p}')-f_{\rm e}(\boldsymbol{p})\right]\left[\rho_{s_0}^{s''}(\boldsymbol{k})T_{s''s'}^{s\;s_0'}\rho_s^{s'}(\boldsymbol{k}') + T_{s_0\,s'}^{s''s}\rho_{s''}^{s'}(\boldsymbol{k}')\rho_s^{s_0'}(\boldsymbol{k})\right]\Big\}.\end{aligned} \tag{99}$$

This equation can be easily rewritten in a more customary for astrophysics form in terms of the Stokes parameters.

4.6. Kinetic Equation in Terms of the Stokes Parameters

The transformation from the polarization matrices to the Stokes parameters is done with the aid of the formulae

$$\left\{\rho_s^{s'}\right\} = \left\{\begin{array}{cc} n_{\rm I}+n_{\rm Q} & n_{\rm U}-in_{\rm V} \\ n_{\rm U}+in_{\rm V} & n_{\rm I}-n_{\rm Q}\end{array}\right\}, \tag{100}$$

where $n_{\rm I} = \rho_s^s(\boldsymbol{k})/2 = \rho(\boldsymbol{k})$ is the mean occupation number of photon states, $n_{\rm Q}$, $n_{\rm U}$, $n_{\rm V}$ are the corresponding characteristics of photon polarization which can be transformed to the standard Stokes parameters (having dimension of intensity) by multiplying them by the factor $2k^3c/(2\pi\hbar)^2$. The equation for polarized radiation interacting with unpolarized electrons takes the following form in terms of the Stokes parameters

$$\begin{aligned}\underline{k}\,\underline{\nabla}\mathbf{n}(\boldsymbol{k},\boldsymbol{r},t) \;=\;& \frac{r_0^2}{2}m^2c^2\int\frac{\mathrm{d}^3k'}{k'}\frac{\mathrm{d}^3p}{p_0}\frac{\mathrm{d}^3p'}{p_0'}\delta(\underline{p}+\underline{k}-\underline{p}'-\underline{k}')\left\{f_{\rm e}(\boldsymbol{p}')[1-(2\pi\hbar)^3 f_{\rm e}(\boldsymbol{p})/2]\mathbf{L}(-\chi)\mathbf{F}\mathbf{L}(\chi')\mathbf{n}(\boldsymbol{k}')\right. \\ +\;& \left.[f_{\rm e}(\boldsymbol{p}')-f_{\rm e}(\boldsymbol{p})]\mathbf{N}(\boldsymbol{k})\mathbf{L}(-\chi)\mathbf{F}\mathbf{L}(\chi')\mathbf{n}(\boldsymbol{k}') - f_{\rm e}(\boldsymbol{p})[1-(2\pi\hbar)^3 f_{\rm e}(\boldsymbol{p}')/2][F+\mathbf{L}(-\chi)\mathbf{A}\mathbf{L}(\chi)]\mathbf{n}(\boldsymbol{k})\right\},\end{aligned} \tag{101}$$

where

$$\mathbf{N}(\boldsymbol{k}) = \begin{pmatrix} n_{\rm I} & n_{\rm Q} & n_{\rm U} & n_{\rm V} \\ n_{\rm Q} & n_{\rm I} & 0 & 0 \\ n_{\rm U} & 0 & n_{\rm I} & 0 \\ n_{\rm V} & 0 & 0 & n_{\rm I} \end{pmatrix}, \quad \mathbf{n}(\boldsymbol{k}) = \begin{pmatrix} n_{\rm I} \\ n_{\rm Q} \\ n_{\rm U} \\ n_{\rm V} \end{pmatrix}, \quad F = \mu_0^2 - 1 + B, \tag{102}$$

$$\mathbf{F} = \begin{pmatrix} F & \mu_0^2 - 1 & 0 & 0 \\ \mu_0^2 - 1 & \mu_0^2 + 1 & 0 & 0 \\ 0 & 0 & 2\mu_0 & 0 \\ 0 & 0 & 0 & B\mu_0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & \mu_0^2 - 1 & 0 & 0 \\ \mu_0^2 - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{L}(\chi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\chi & \sin 2\chi & 0 \\ 0 & -\sin 2\chi & \cos 2\chi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{103}$$

The matrices $\mathbf{L}(\chi)$ have the same properties as matrices (90). They are commutative and their transposition is equivalent to the inversion or changing the sign of the argument. Note that the same argument χ appears with the opposite signs in matrices $\mathbf{L}$ (around matrix $\mathbf{A}$) in the last term of equation (101).

As we have mentioned above, the equation of the form (101) for non-degenerate electrons (not accounting for the exclusion principle) was given in Nagirner (1994). The term with matrix $\mathbf{A}$ was omitted there. It can be easily shown that this term disappears if the electrons have an isotropic distribution (even if they are degenerate). In the major part of Nagirner's paper, the isotropic electron distribution was assumed, so that this omission did not introduced any errors. [1] The matrix $\mathbf{A}$ was omitted also in papers of the authors (Nagirner & Poutanen 1993, 1994) where the rhs of equation (101) was averaged over the directions of electron momenta and the five functions describing the redistribution of radiation in frequency, angles and polarization states were obtained. However, these papers were devoted to the isotropic electrons as well.

Electron isotropy means that the medium is locally isotropic, and therefore the attenuation (i.e. the terms corresponding to the last square brackets in eq. [101]) is described not by the matrix, but only by the scalar determined by the Klein—Nishina cross-section, F. The cross-section averaged over momentum of scattered photons can be found in Nagirner & Poutanen (1994). Hence we can conclude here that polarized electrons can influence both the linear and circular polarization by means of attenuation through scattering. Unpolarized, even non-isotropic, electrons can introduce and change in this process only linear polarization.

5. Conclusions

In this paper, we have deduced the relativistic kinetic equations which describe the behavior of the rarefied photon and electron (or positron) gases interacting with each other via Compton scattering. We accounted here for stimulated effects for the photons and for the exclusion principle for the electrons. We considered arbitrary polarization states of photons and electrons. We presented also the kinetic equation for polarized photons scattered by unpolarized electrons in terms of the Stokes parameters. The expressions for the scattering amplitudes and cross-sections are derived simultaneously. There are no limitations on photon and electron energies.

Note that all the deductions were made by means of relativistic quantum electrodynamics methods and all the equations obtained are relativistically covariant. For particular scattering problems they, of course, must be adapted to the geometry and symmetries of a medium and initial and boundary conditions (see e.g. Nagirner & Poutanen 1993).

Finally, we would like to notice that the factors $1 + \rho(\boldsymbol{k})$ and $1 - (2\pi\hbar)^3 f_{\rm e}(\boldsymbol{p})/2$ accounting for the induced effects for photons and the exclusion principle for electrons in the scalar kinetic equations, should be replaced by the new factors $\delta_s^{s'} + \rho_s^{s'}(\boldsymbol{k})$ and $\delta_\tau^{\tau'} - \rho_\tau^{\tau'}(\boldsymbol{p})$ in the kinetic equations for polarized photons and electrons. This rule can help to formulate kinetic equations for more complex processes for which a direct derivation could be very complicated. The processes in strong magnetic fields can serve as such examples.

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[1] Let us note here that the generalization of the Babuel-Peyrissac—Rouvillois equation to polarized radiation (i.e. a limiting version of eq. [101] for weakly relativistic Maxwellian electrons and soft photons) was deduced in Nagirner (1994). The English version of the paper contains many misprints in the generalized equation. In the Russian version, $\tilde{n}$ should be substituted instead of n in the first row of his equation (41) and the brackets before $\tilde{n}_1''$ and after $\tilde{n}_1'$ in the third row should be removed. Hansen & Lilje (1999) who used the original equation of the form (101) presented in Nagirner (1994) also noted the aforementioned misprints.

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