Numerical Simulations of Dynamos Associated with ABC Flows

Vasilis Archontis and Bertil Dorch[1]

The Astronomical Observatory, The Niels Bohr Institute, Juliane Maries Vej 30, DK-2100 Copenhagen Ø, Denmark

Abstract. We present results from a series of numerical simulations of kinematic dynamo action by spatially periodic ABC flows. Numerical results are reported for the 'normal' ABC flow with $A : B : C = 1 : 1 : 1$ and for the case of $A : B : C = 5 : 2 : 2$ without stagnation points. We focus on the dynamics of the magnetic structures that are developed by the flow and outline the dynamical processes that are responsible for the amplification of the magnetic field.

1. Introduction

The general dynamo problem refers to the amplification and maintainance of a magnetic field in an electrically conducting fluid of resistivity η. In the kinematic case the velocity field $\mathbf{u}$ as given, and not influenced by the development of the magnetic field. Thus, the Lorentz force is assumed to be negligible in the equation of motion. The governing equation for the magnetic field is the familiar induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \tag{1}$$

We study the kinematic problem for steady three-dimensional flows with and without stagnation points, namely the special class of ABC flows. The velocity field of an ABC flow is given by the sum of three parameterized Beltrami waves:

$$\mathbf{u} = A(0, \sin kx, \cos kx) + B(\cos ky, 0, \sin ky) + C(\sin kz, \cos kz, 0), \tag{2}$$

where A, B and C are constant coefficients and k is the wavenumber of the flow. The numerical simulations are performed in a $\ell = 2\pi$-periodic box with $k = 1$ with a computational code that is a simplified version of the finite difference staggered mess code by Nordlund, Stein and others (see e.g. Nordlund et al. 1994).

The flow $\mathbf{u}$ is called a *fast dynamo* if the exponential growth rate of the magnetic field remains positive and bounded away from zero in the limit of vanishing diffusion ($\eta \to 0$). This means that the magnetic field amplified by a fast dynamo evolves on a timescale comparable to the advective timescale. If

[1] Present address: Stockholm Observatory, SE-133 36 Saltsjöbaden, Sweden

the growth rate tends to zero then the dynamo is a *slow dynamo* and the growth of the field occurs on a timescale between that of advection and the much longer diffusive timescale.

The 'normal' ABC flows with $A : B : C = 1 : 1 : 1$ (also referred to as the "111" case) is probably the most studied type of ABC flow in the literature. The reason is that it, on the one hand, is a simple example of how a complex field may be able to amplify a weak seed magnetic field and, on the other hand, displays a behavior that resembles real astrophysical dynamos.

Results for the normal ABC flow together with some aspects of the kinematic dynamo produced by this flow are discussed in Section 2. In Section 3 preliminary results are presented for the more general case of ABC dynamo action without stagnation points, pointing out similarities and differences compared to the normal case. Section 4 is the conclusion.

2. The Normal ABC Flow

In the case of normal ABC flows two windows of dynamo action exist: Arnold and Korkina (1983) identified a window of dynamo action ranging from a magnetic Reynolds numbers $\mathrm{Re_m} = u\ell/\eta$, of 8.9 to 17.5 just above which the magnetic field decayed. Galloway & Frisch (1986) discovered another window of dynamo action beginning at $\mathrm{Re_m} = 27$ and extending at least beyond 550. In both windows cigar-like magnetic structures appear that are centered on *stagnation points* of the flow (see below) as found by e.g. Childress (1979), Childress & Soward (1985) and Galloway & Frisch (1986).

Even though the normal ABC flow is steady, in certain regions of the flow trace particles follow chaotic paths, but more relevant for the modes of dynamo action in the general diffusive case is the stretching ability of the flow.

2.1. Flow Topology

The normal ABC flow has 8 stagnation points (see Figure 1) of two different types;

- α-type stagnation points where stream lines are diverging along an axis through the stagnation point, and converging in the plane perpendicular to the axis,
- β-type stagnation point where stream lines are converging along the axis, and diverging in the plane.

The stream lines of the flow have a three-fold symmetry in the converging (diverging) planes through the α (β) type stagnation points (see Figure 1). The three-fold symmetric 'leaves' of the converging/diverging stream lines are separated by separator lines; both these separator lines as well as the leaves of convergence/divergence connect to other stagnation points: While the separator lines and leaves of diverging stream lines of a β point connect to three α type stagnation points, the reverse is true for an α point.

In itself it is not so important that there happens to be 8 stagnation points in the normal ABC flow; after all, many interesting flows such as the ones we discuss in the latter part of this paper do not possess stagnation points, and in

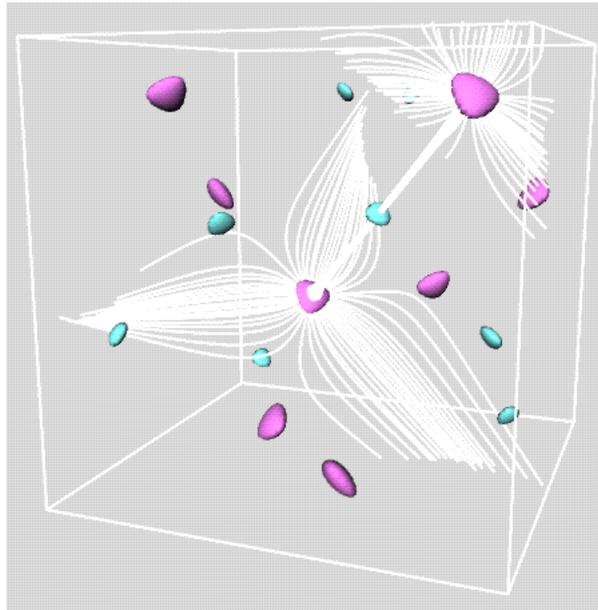

Figure 1. A view showing the structure of the normal ABC flow: The small bead-shaped isosurfaces show the positions of the stagnation points of the flow (α type and β type). The stream lines illustrate the flow topology. Near the center of the view is a β type stagnation point in a plane of three-fold symmetry with diverging stream lines. In the upper right corner is an α type stagnation point connecting to the β type point via a heteroclinic orbit.

any case a stagnation point may be removed by a translation of the coordinate system. Rather, it is the stretching ability of the flow that is relevant for the dynamo action. In the high degree of symmetry of the normal ABC flow, the stagnation points coincide with local extrema of the stretching rate. Hence, the two types of stagnation points may be seen as convenient markers of these regions.

2.2. Exponential Amplification

As shown in Figure 2, an initially weak seed field is amplified on an exponential timescale. For intermediate $\mathrm{Re_m}$ (in the second window of dynamo action), an oscillating behavior is associated with the growth (as also noted by e.g. Galloway et al. 1986, Galanti et al. 1992 and Galanti et al. 1993). The period of the oscillation may be understood as a direct consequence of the spatial periodicity of the ABC flow (Galanti et al. 1992). The period of the oscillation increases with increasing $\mathrm{Re_m}$ until a transition to a non-oscillating regime occurs at a $\mathrm{Re_m}$ of about 200 (Lau & Finn 1993), see Figure 2. As mentioned by Galloway et al. (1992) and Childress & Gilbert (1995) it is possible to rig the initial seed magnetic field so that only one mode of dynamo action is present in the calculations by choosing the following special initial magnetic field:

$$\mathbf{B} = (\sin(kz) - \cos(ky), \sin(kx) - \cos(kz), \sin(ky) - \cos(kx)), \qquad (3)$$

that is an eigenmode in the diffusion-less case ($\eta = 0$).

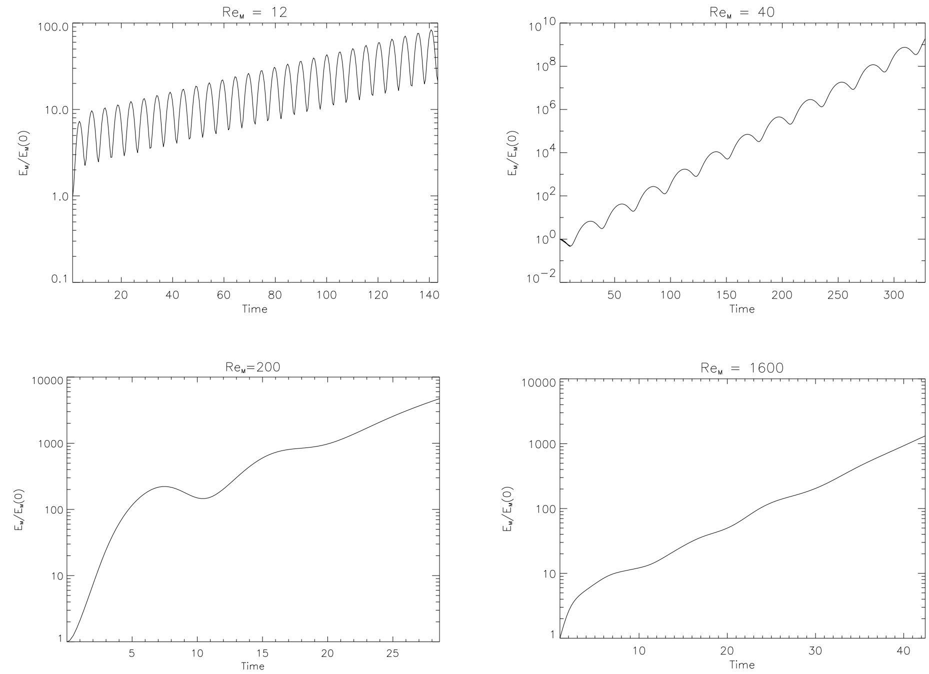

Figure 2. Four panel showing the magnetic energy as function of time for increasing magnetic Renolds number: $\mathrm{Re_m} = 12$, 40, 200, and 1600.

In that case the magnetic field is not amplified in the second window of dynamo action (see also Galloway et al. 1992). Childress & Gilbert (1995) used this initial condition together with the so called 'flux conjecture' to try to deduce the limiting growth rate of the ABC flow. They found exponential growth of the flux in a selected region. The growth rate derived does not, however, agree with the asymptotic growth rates found by Galloway et al. (1992) and Lau & Finn (1993) and one should indeed not expect to recover the growth rate of an exponential growing mode, by studying the stretching of field lines in a secularly decaying mode. Figure 3 shows that the mode in the case of the initial condition given by Eq. 3 is an exponentially decaying oscillating mode. Because of numerical round-off errors the amplitude of the growing mode is not identically equal to zero in the initial condition, and eventually its inevitable growth and the decay of the initialized mode results in a transition from decay to growth of the total magnetic energy. Galloway et al. (1992) found no growing solution and they concluded that "something odd is going on".

2.3. Dynamical Amplification Process

When the induction equation is evolved from a weak uniform seed field, flux "cigars" rapidly arise at four of the eight stagnation points; the α-points. The flux cigars are aligned along the axis of divergence through the α type stagnation points, and point directly to the β type points.

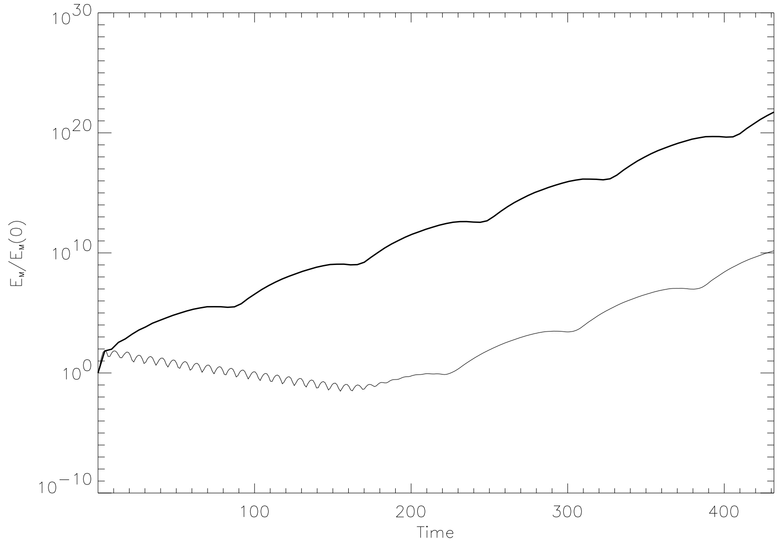

Figure 3. The evolution of the total magnetic energy E_M as a function of time for two experiments (with the same moderate $\mathrm{Re_m}$) with an initially uniform magnetic field (thick line) and with the initial condition given by Eq. 3 (thin line).

We find that a second set of flux cigars is formed next to the 4 primary cigars, and conclude that rather than "obscuring" the physics (as noted by Galloway & O'Brian 1993), these structures are absolutely essential. These secondary cigars have the opposite polarity of the neighboring cigars, and they are connected by reconnecting field lines to the primary cigars.

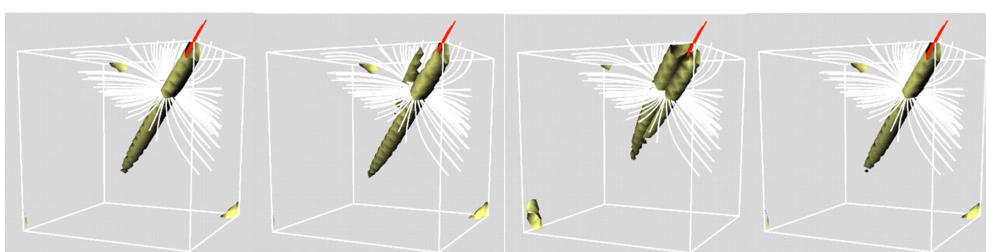

Figure 4. Four snapshots (only part of the periodic box is shown) of an experiment within the second window of $\mathrm{Re_m}$ showing the evolution of the double flux cigars. Also shown are converging stream lines and the diverging axis through the α point. The four pictures correspond to four instants during the oscillation in the magnetic energy, at fractions of 0, 0.14, 0.87 and 1 of the period.

These secondary flux cigars form near the α point at the separator line next to the primary cigar (see Figure 4). As the secondary cigar forms, the primary cigar moves slightly away from the center of the stagnation point. In the simulation within the oscillatory regime (in the second window of dynamo action), as the energy increases at the beginning of a 'cycle', the secondary cigar increases in size and field strength until the two cigars become equal in size midway through the cycle. The primary cigar now becomes smaller and actually vanishes at the end of the cycle, and at that time the growth of the energy slows down, and even reverses (see Figure 2). At that point the flux cigar that was previously the secondary cigar moves into the center of the stagnation point (see the evolution in Figure 4).

It is possible to illustrate the whole cycle by considering the "path" of magnetic field lines (Dorch 1998): In the beginning of a cycle (say, at a local energy minima in Figure 2), there are only one cigar at each stagnation point, but field lines begin to pile up at the separator line close to the α point where a secondary cigar eventually forms. These field lines come from the plane of divergence of one of the neighboring β points and move along two sets of converging stream lines on opposite sides of the separator line; where the stream lines reach the α point, they have twisted so that field lines that are carried along them are parallel to the axis of divergence through the α point. At the opposite side of the α point, one of the three leaves of diverging stream lines from a β point supply field lines of the right polarity to the primary flux cigar. While these two flux cigars sit near the α point and receive field lines, the field lines that forms them may reconnect between them because of their opposite polarities and the non-vanishing diffusivity. The reconnected field lines moves as tight "hooks" out along the axis of divergence through the α point towards the plane of the β point where an intricate folding takes place and the field lines are stretched out into a triangular shape along the three fold symmetric leaves, before being carried to yet another α point (Dorch 1998).

The magnetic energy grows through most of the cycle, but the main growth takes place while the two cigars have about equal sizes. This is the time at which the most rapid reconnection take place, and thus the time where the largest amount of magnetic flux is released down along the axis of divergence through the α points. Thus, the reconnection process is essential to the operation of this mode of the dynamo: If there were no reconnections, the magnetic field near the β points could not be replenished. The β point plane of divergence constitutes a discontinuity region, where field lines change direction from above to below.

The cycle discussed above is actually only half a cycle; a full cycle requires that the flux cigars return to the original polarity and this is only achieved after two of the above cycles.

For large $\mathrm{Re_m}$ (within the non-oscillatory regime) the amplification process is similar to what was described above for the lower $\mathrm{Re_m}$ case. However, for large $\mathrm{Re_m}$, there are no oscillation associated with the amplification process; the double cigars always consist of one large strong cigar and one smaller and weaker one with the opposite polarity. The secondary cigars never becomes stronger than the primary, but the mechanism that drives the amplification process remains the same for increasing $\mathrm{Re_m}$.

3. ABC Flows without Stagnation Points

An interesting question is in what respect the evolution of the magnetic field in ABC flows without stagnation points differs from the "standard" case. We have experimented with three such cases, namely the A:B:C=5:2:2, A:B:C=4:1:1 and the A:B:C=2:1:1 case, and show preliminary results from our numerical experiments.

The time evolution of the magnetic energy for all three cases is shown in Figure 5: Initially, the dominating mode is an oscillatory exponentially growing mode, but after a large number of oscillations, a new dominating mode takes over the magnetic field amplification. This mode corresponds to a growth rate that is more than twice as large as the growth rate of the first mode and it is also oscillatory, but with a 5 times smaller period and a very small amplitude.

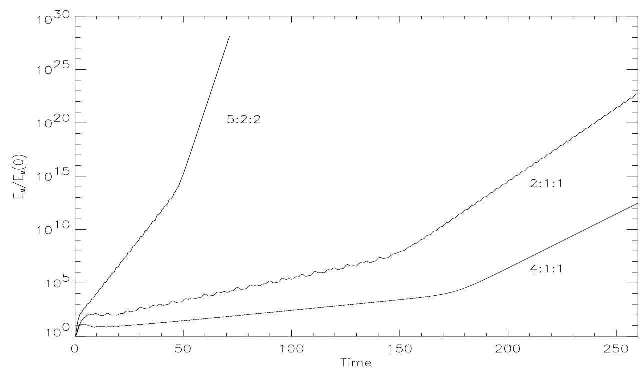

Figure 5. The total magnetic energy as a function of time on a logarithmic scale for the 5:2:2 case (left), 2:1:1(middle) and 4:1:1(right) at $\mathrm{Re_m} = 120$.

3-D visualizations of the magnetic field reveal that initially double magnetic flux sheets develop (see Figure 6). These sheets have oppositely aligned fields and current sheets form in the gap between them. Reconnection occurs in the gap and the reconnected field lines form tight “hooks” as they pass through the current sheet (see Figure 7, left panel).

Figure 7 (left panel) shows that the strongly bent field lines form two different pairs of flux sheets (see also Figure 6, right panel). A visualization of strong current density isosurfaces is shown in the right panel of Figure 7, pointing out the location of the current sheets between the oppositely aligned sets of field lines. The picture outlined in the last paragraphs allow us to understand the amplification process that maintains the initially oscillatory growth of the magnetic field: Folding of the field takes place and the magnetic energy increases. However, this folding is not constructive. The field is not brought into perfect alignment and there are bands of field pointing in the opposite direction in between the sheets. Again a key ingredient for the above described mechanism is the reconnection of field lines. If there was no reconnection, the magnetic field could not be replenished in the regions where folding of the field occurs and the

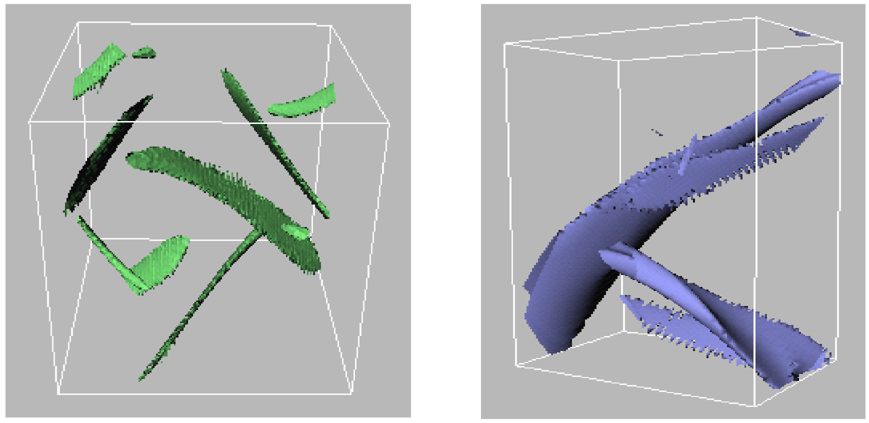

Figure 6. The initial magnetic topology in the A:B:C=5:2:2 case (left). Isosurfaces of magnetic field strength are visualized, showing structures that are sheet-like. The right panel shows that double sheets are created after the initial transient phase (only a small part of the computational box is shown). These structures have oppositely aligned field lines.

amplification process of this mode would not be possible. Thus, this process is similar to the amplification process in the normal ABC case. The main differences are that double sheets supply field lines by reconnection instead of double cigars, and that the flux structures are advected instead of being confined to the regions around the stagnation points.

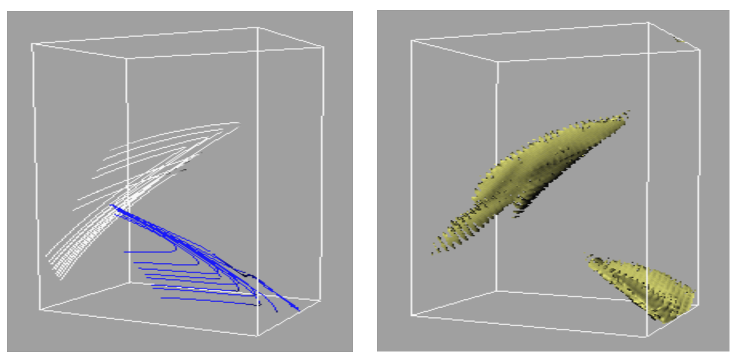

Figure 7. Two different sets of magnetic field lines (left). The right panel contains isosurfaces of strong electric current showing the relation of the current to the double sheets (5:2:2 case).

The operation of the high growth rate mode in Figure 5, is somewhat different: The magnetic field continues to consist of sheet-like structures but these structures now have the same polarity. Thus, the flow brings magnetic field lines of the same polarity into alignment, and constructive folding takes place increasing the magnetic energy significantly.

Figure 8 shows three slices of the xy-plane of the computational box. There are fine bands (sheets) of strong magnetic field, indicating the stretching by the flow. Two of the sheets on the right edge of the first slice have the same sign and point roughly in the same direction. On the bottom and right of the same slice, however, there are sheets of field pointing in the opposite direction. As time

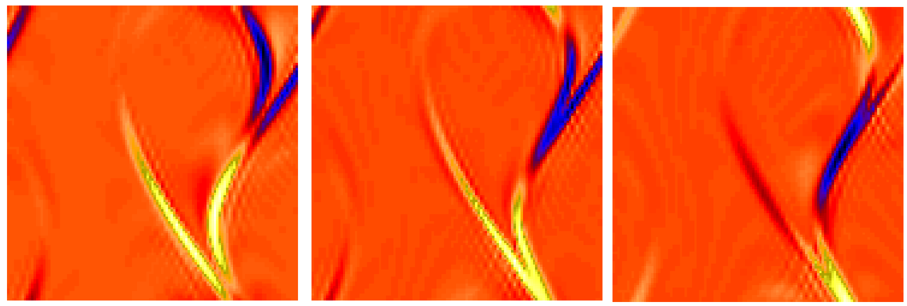

Figure 8. The three panels show the constructive folding of the magnetic field (5:2:2 case, $\mathrm{Re_m}$=120). The z-component of the magnetic field is plotted for $0 \leq x, y \leq 2\pi$.

goes on, both pairs (in black and white color) come very close together and fold. Thus the field is brought into perfect alignment by the flow and the folding is constructive. Each time the sequence is repeated, the flux is increased because of the constructive reinforcement of the sheet-like magnetic field structures.

4. Conclusions

The results obtained so far concern kinematic dynamo action produced by different types of ABC flows. In the normal $A : B : C = 1 : 1 : 1$ case, we find that the dynamo action in the two windows of $\mathrm{Re_m}$ correspond to two distinct modes. In both cases the replenishing of the field near the β points is crucial for the operation of the dynamo.

Certain properties of the magnetic transport are nearly invariant as $\mathrm{Re_m}$ is increased; the size of the regions where diffusion is important become smaller as $\mathrm{Re_m}^{-\frac{1}{2}}$ but reconnection still takes place and the field in the crucial β regions continues to be replenished. In the bulk of the flow where the stretching takes place, the decrease of the diffusivity is unimportant since the field lines there are not influenced by diffusion; they tend to obtain a certain alignment with the flow topology given by the stretching that is an invariant property, and hence the exponential rate of increase the magnetic field remains nearly the same. It thus appears very unlikely that the double-cigar mode should go away in the limit of infinite $\mathrm{Re_m}$, and that the normal ABC flow would not be a fast dynamo.

In the case of an ABC flow without stagnation points (the $A : B : C = 5 : 2 : 2$ case), we show that the magnetic structures are sheet-like rather than cigar-like. There is one mode where the amplification process is similar to the case of the normal ABC flow while a second mode based on constructive folding has a larger exponential growth rate and thus becomes the dominating mode.

Similar behavior is found for other flows without stagnation points (e.g. for $A : B : C = 2 : 1 : 1$ and $A : B : C = 4 : 1 : 1$) and the magnetic structures that are developed have a similar topology. Reconnection and stretching of field lines are key ingredients for the amplification of the field, but the results for this type of flow are preliminary. Numerical simulations, at $\mathrm{Re_m}$ up to 800, have been performed and show that the growth rate increases with $\mathrm{Re_m}$. Thus, there is evidence that these flows are fast dynamos as well, though simulations at higher $\mathrm{Re_m}$ would be of great interest and necessary to confirm any conclusion.

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